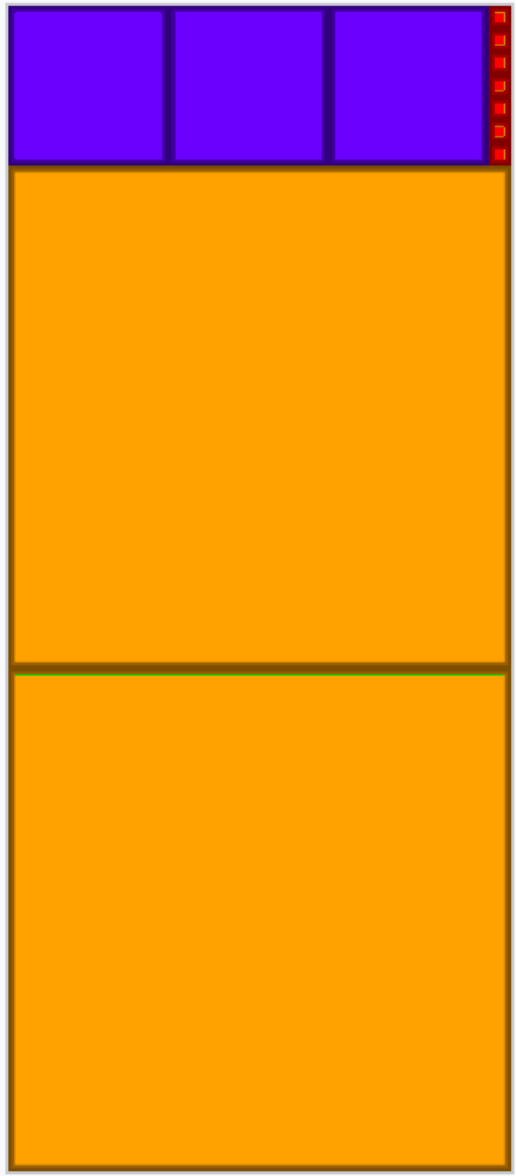
Euclidean Algorithm

For gcd (greatest common divisor)

- https://en.wikipedia.org/wiki/Euclidean_algorithm

Subtraction-based animation of the Euclidean algorithm. The initial rectangle has dimensions $a = 1071$ and $b = 462$. Squares of size 462×462 are placed within it leaving a 462×147 rectangle. This rectangle is tiled with 147×147 squares until a 21×147 rectangle is left, which in turn is tiled with 21×21 squares, leaving no uncovered area. The smallest square size, 21, is the GCD of 1071 and 462.



```
function gcd(a, b)
    while a ≠ b
        if a > b
            a := a − b
        else
            b := b − a
    return a
```

```
function gcd(a, b)
    if b = 0
        return a
    else
        return gcd(b, a mod b)
```

Extended Euclidean Algorithm

- https://en.wikipedia.org/wiki/Extended_Euclidean_algorithm
- https://crypto.stackexchange.com/questions/5889/calculating-rsa-private-exponent-when-given-public-exponent-and-the-modulus-fact

Your goal is to find d such that $ed \equiv 1 \pmod{\varphi(n)}$.

Recall the EED calculates x and y such that $ax + by = \gcd(a, b)$. Now let $a = e, b = \varphi(n)$, and thus $\gcd(e, \varphi(n)) = 1$ by definition (they need to be coprime for the inverse to exist). Then you have:

$$ex + \varphi(n)y = 1$$

Take this modulo $\varphi(n)$, and you get:

$$ex \equiv 1 \pmod{\varphi(n)}$$

And it's easy to see that in this case, $x = d$. The value of y does not actually matter, since it will get eliminated modulo $\varphi(n)$ regardless of its value. The EED will give you that value, but you can safely discard it.

The following table shows how the extended Euclidean algorithm proceeds with input 240 and 46. The greatest common divisor is the last non zero entry, 2 in the column "remainder". The computation stops at row 6, because the remainder in it is 0. Bézout coefficients appear in the last two entries of the second-to-last row. In fact, it is easy to verify that $-9 \times 240 + 47 \times 46 = 2$. Finally the last two entries 23 and −120 of the last row are, up to the sign, the quotients of the input 46 and 240 by the greatest common divisor 2.

index i	quotient q_{i-1}	Remainder r_i	s_i	t_i
0		240	1	0
1		46	0	1
2	$240 \div 46 = 5$	$240 - 5 \times 46 = 10$	$1 - 5 \times 0 = 1$	$0 - 5 \times 1 = -5$
3	$46 \div 10 = 4$	$46 - 4 \times 10 = 6$	$0 - 4 \times 1 = -4$	$1 - 4 \times -5 = 21$
4	$10 \div 6 = 1$	$10 - 1 \times 6 = 4$	$1 - 1 \times -4 = 5$	$-5 - 1 \times 21 = -26$
5	$6 \div 4 = 1$	$6 - 1 \times 4 = 2$	$-4 - 1 \times 5 = -9$	$21 - 1 \times -26 = 47$
6	$4 \div 2 = 2$	$4 - 2 \times 2 = 0$	$5 - 2 \times -9 = 23$	$-26 - 2 \times 47 = -120$

```
function extended_gcd(a, b)
    (old_r, r) := (a, b)
    (old_s, s) := (1, 0)
    (old_t, t) := (0, 1)

    while r ≠ 0 do
        quotient := old_r div r
        (old_r, r) := (r, old_r − quotient × r)
        (old_s, s) := (s, old_s − quotient × s)
        (old_t, t) := (t, old_t − quotient × t)

    output "Bézout coefficients:", (old_s, old_t)
    output "greatest common divisor:", old_r
    output "quotients by the gcd:", (t, s)
```

On homework 2 and the final, the RSA scheme will already be set up for you with an *e* and a *d* so you won't need to do the extended Euclidean algorithm.