Fast Fourier Transform

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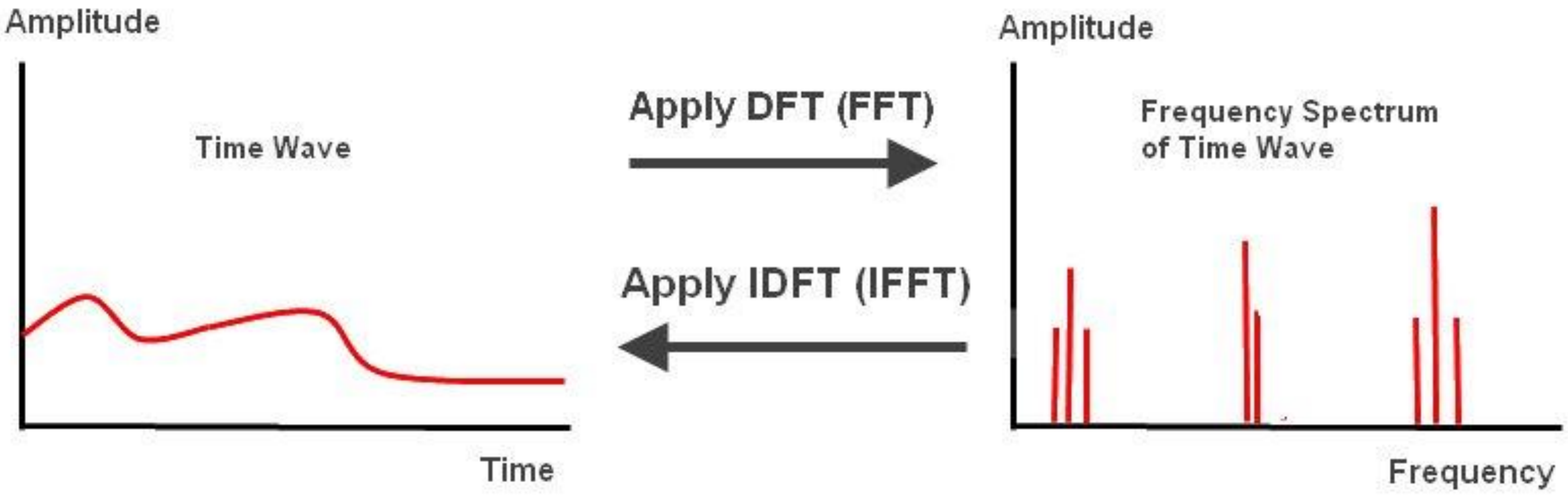
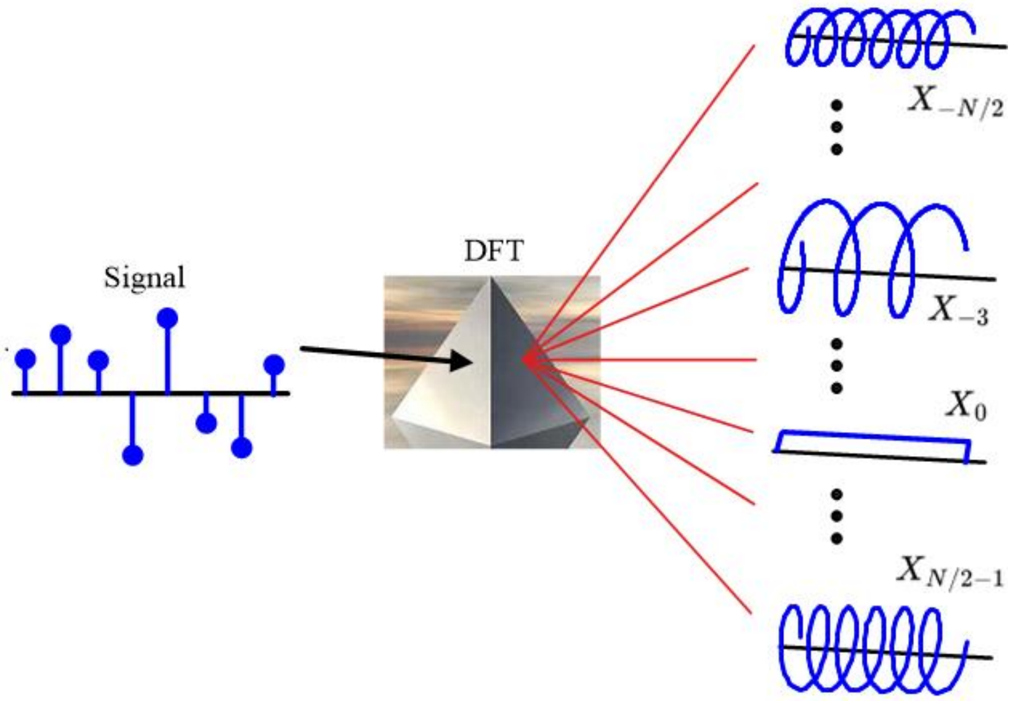
CSE 468





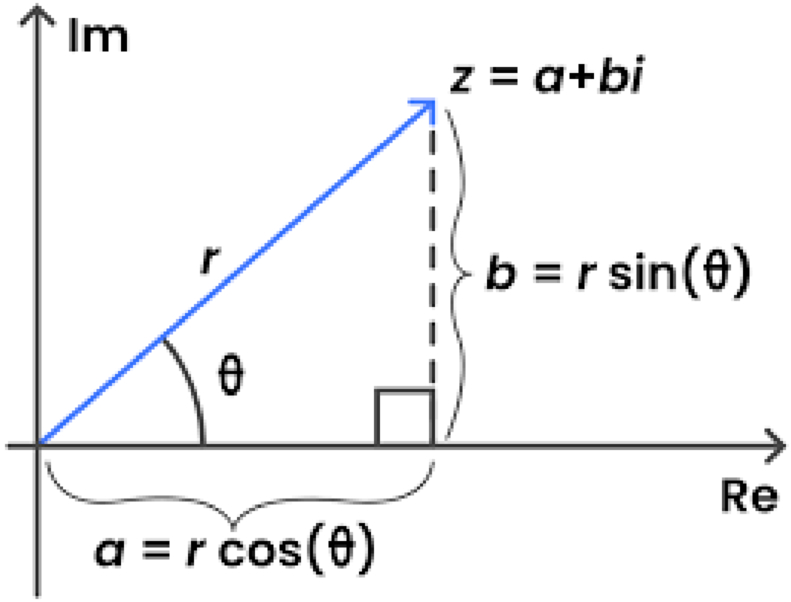
https://www.alwayslearn.com/DFT%20and%20FFT%20Tutorial/DFTandFFT_BasicIdea.html

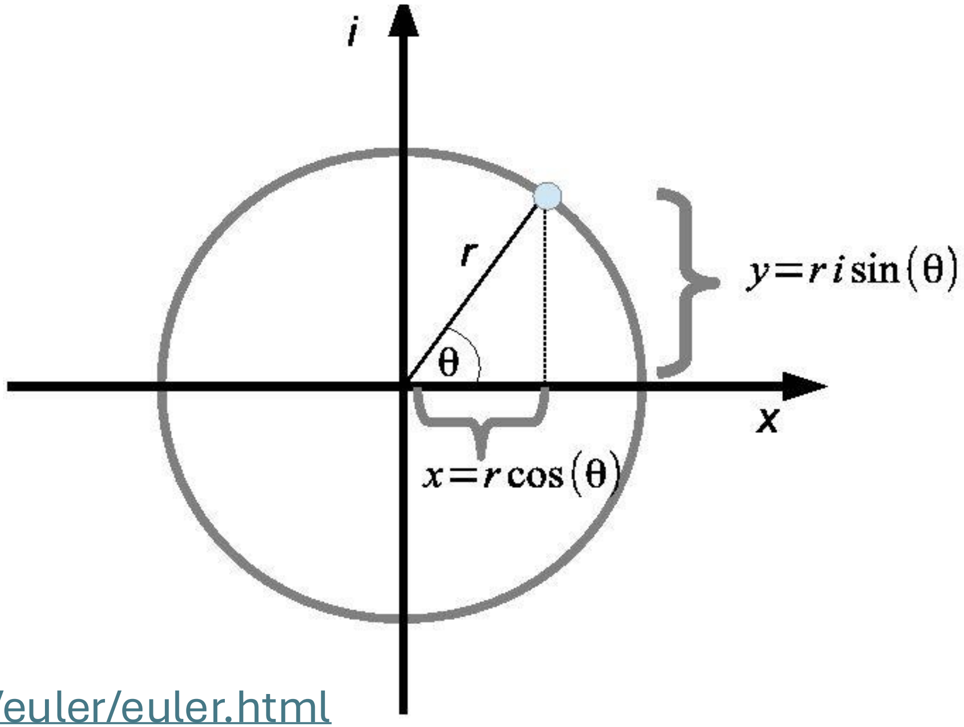
https://wirelesspi.com/the-discrete-fourier-transform-dft/

Complex numbers

$$\begin{aligned} z &= a + ib \\ &= |z| \cdot (\cos\theta + i\sin\theta) \qquad \text{where} \quad |z| = \sqrt{a^2+b^2} \\ &= |z| \cdot e^{i\theta} \qquad\qquad\qquad\qquad \theta = \tan^{-1}\left(\frac{b}{a}\right) \end{aligned}$$

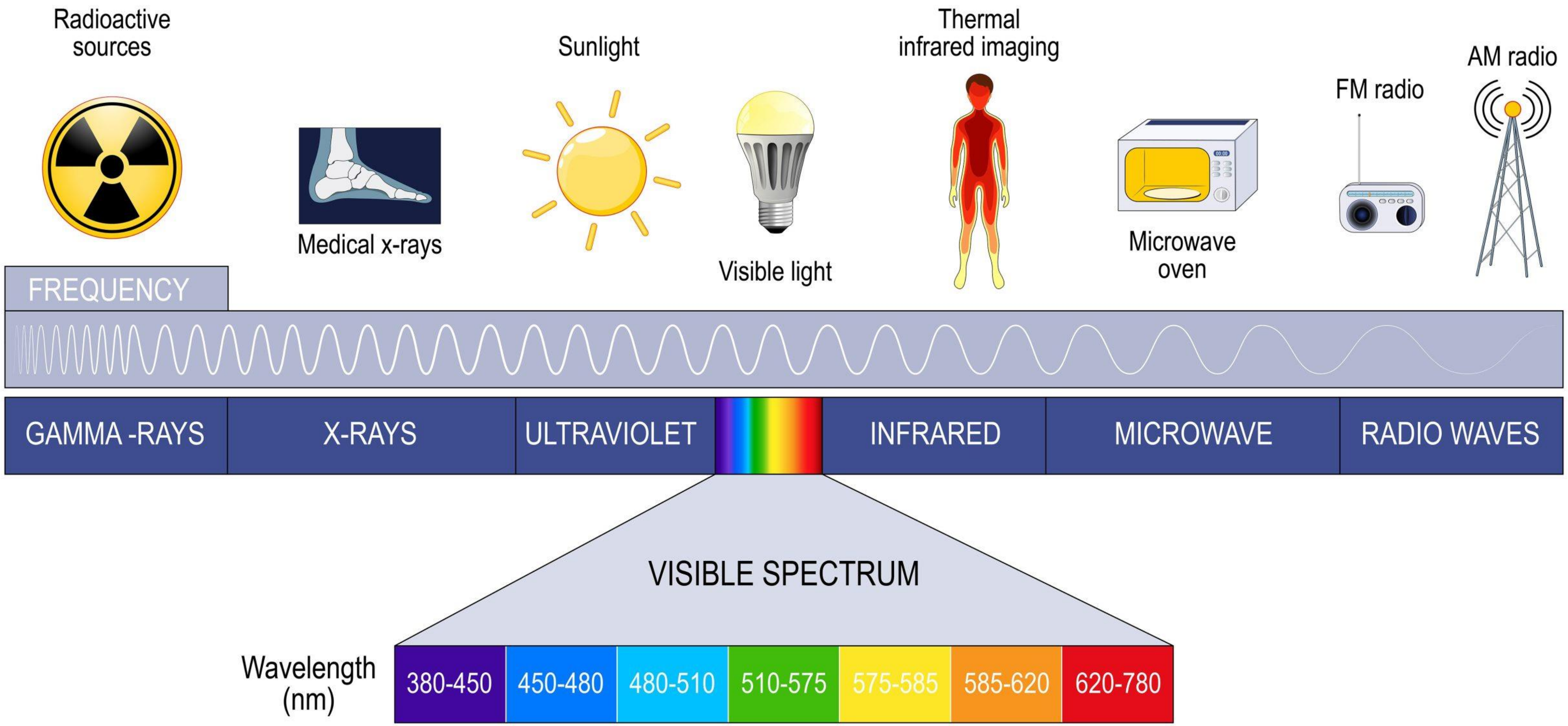
Euler's Theorem: $re^{i\theta} = rcos(\theta) + i \cdot rsin(\theta) \qquad r = |z|$





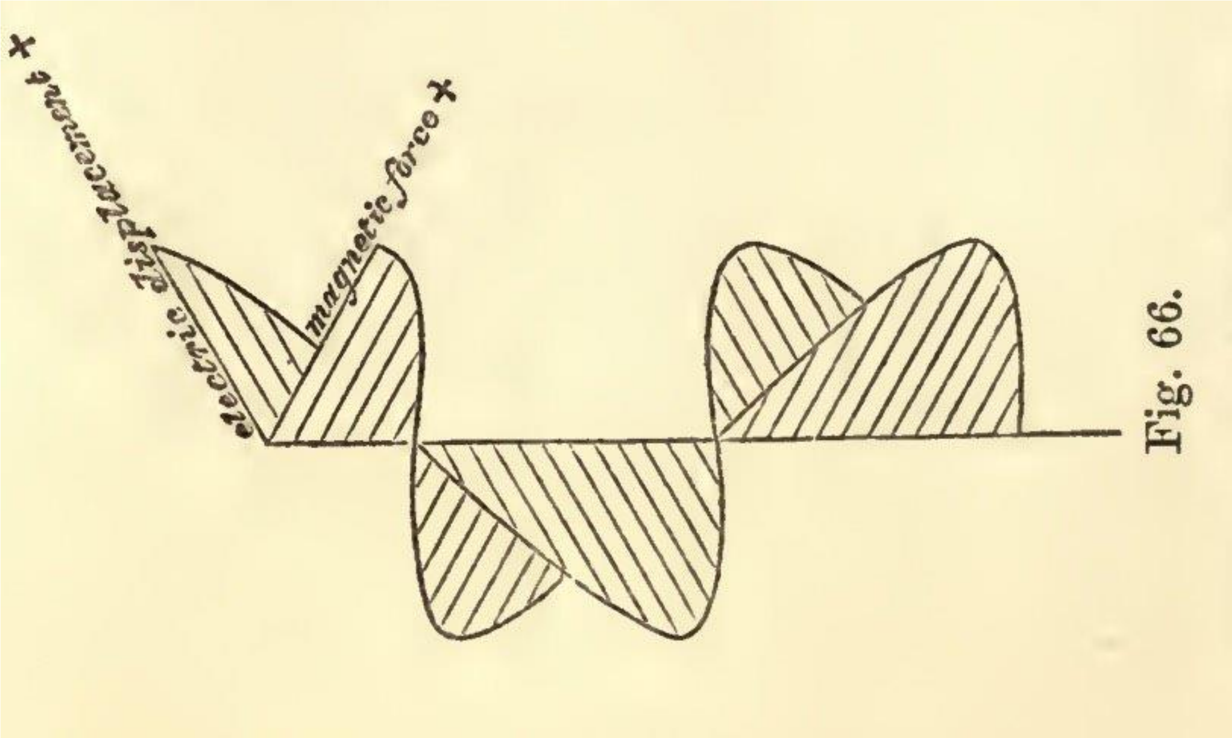
https://www.songho.ca/math/euler/euler.html

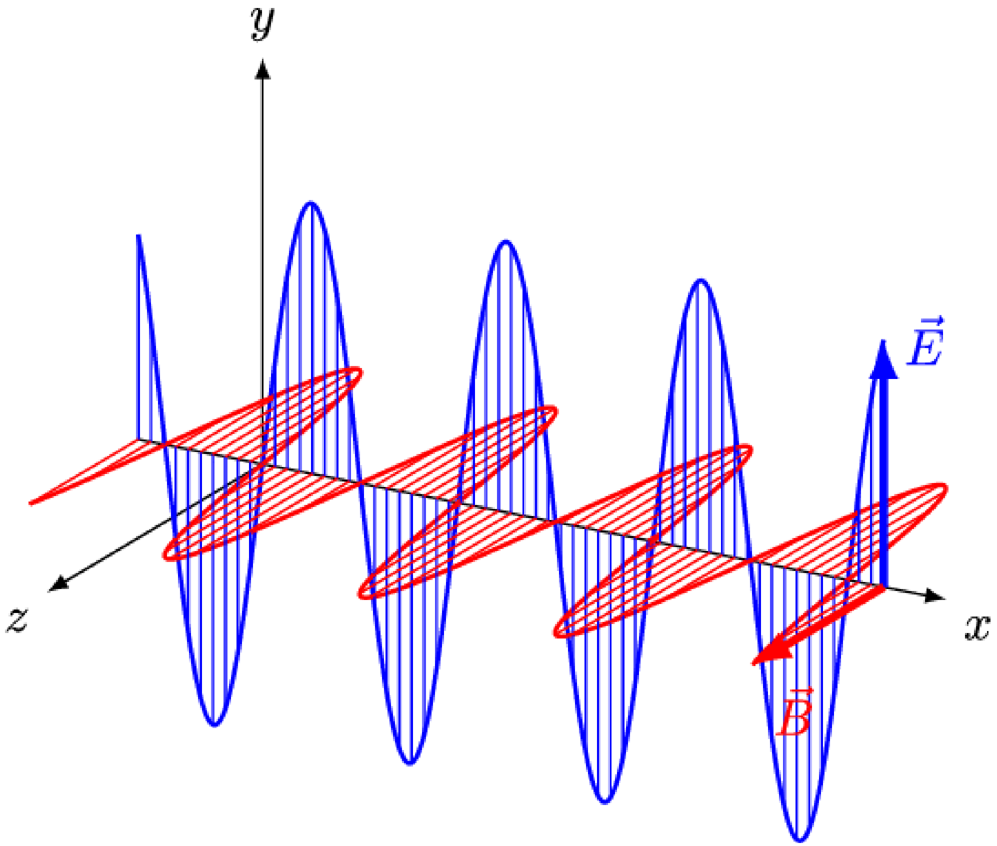
Electromagnetic spectrum



https://www.cdc.gov/radiation-health/about/electromagnetic-spectrum.html

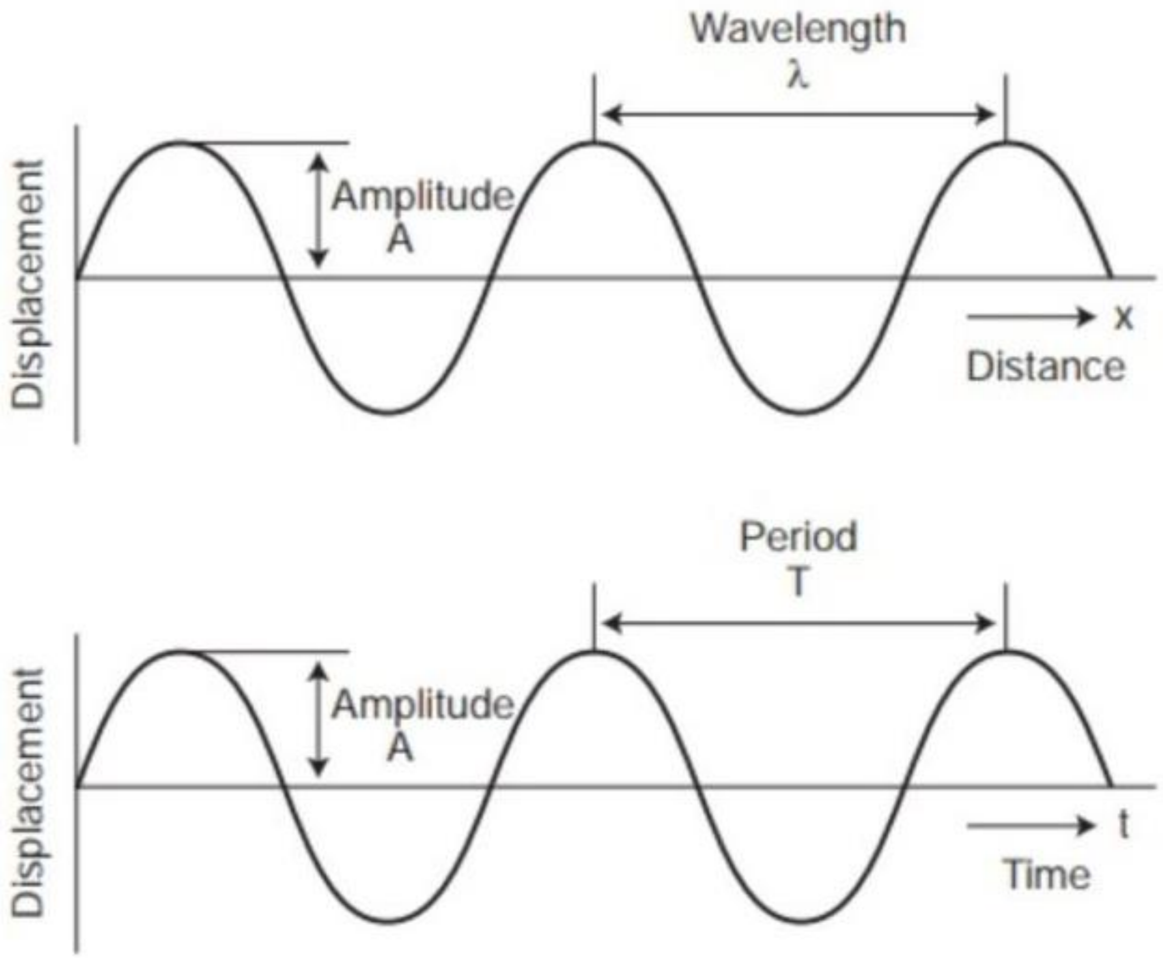
A wave is a disturbance (oscillatations) that travels through a medium or space, transferring energy without transferring matter.

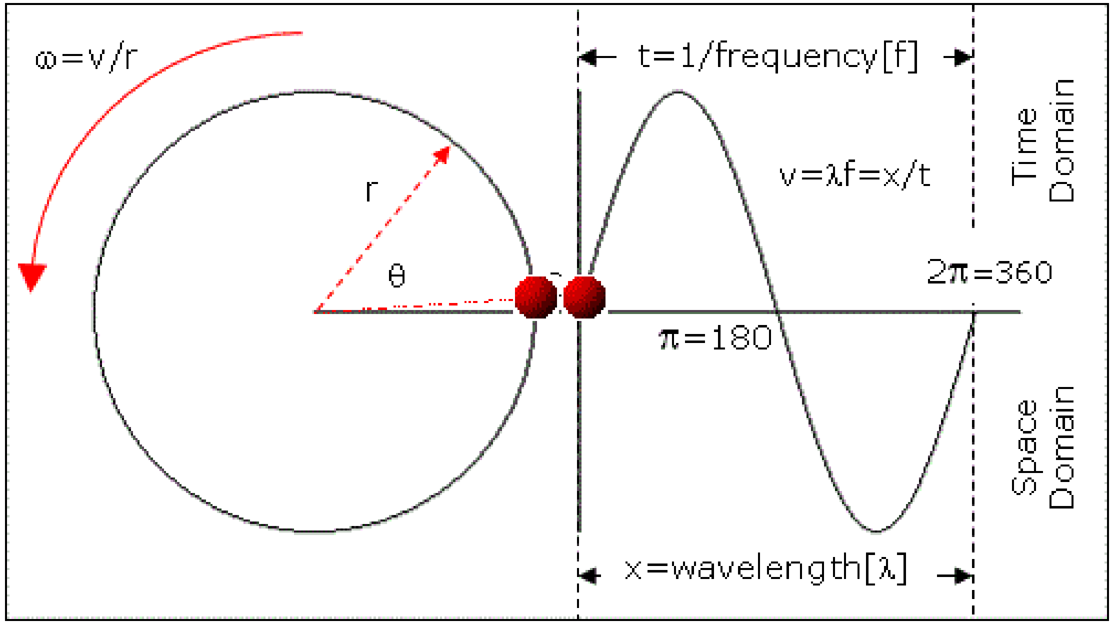




https://sites.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/quantum_theory_complex/complex.html

https://commons.wikimedia.org/wiki/File:EM-Wave.gif

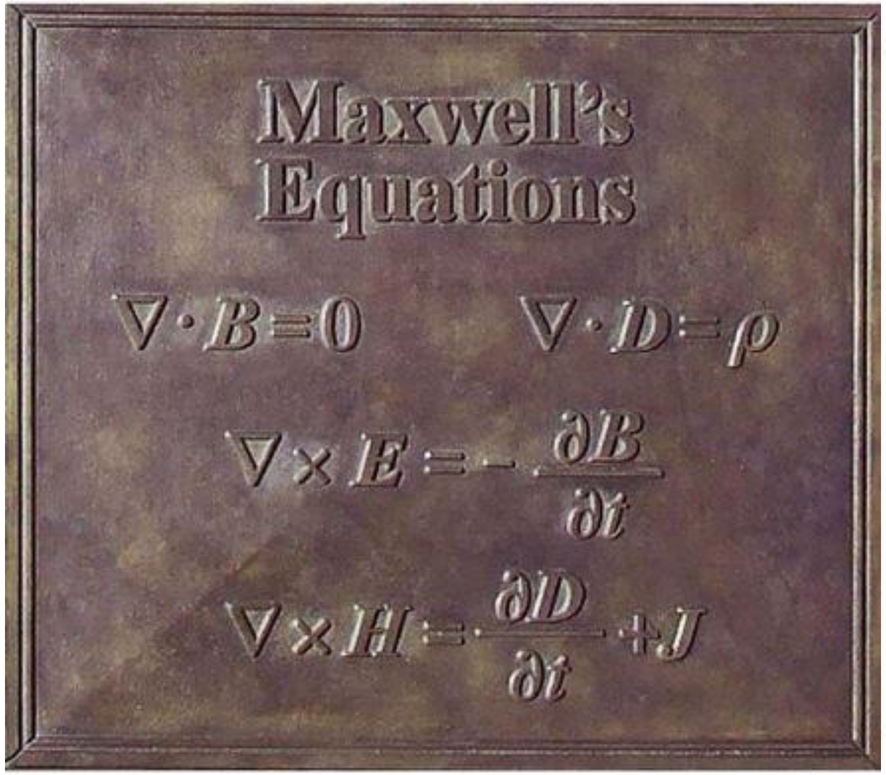




https://blog.stuidapp.com/characteristics-of-sound-waves/

https://turbomass.wordpress.com/2019/05/03/vibration-time-wave-form-single-frequency/

Maxwell's Equations

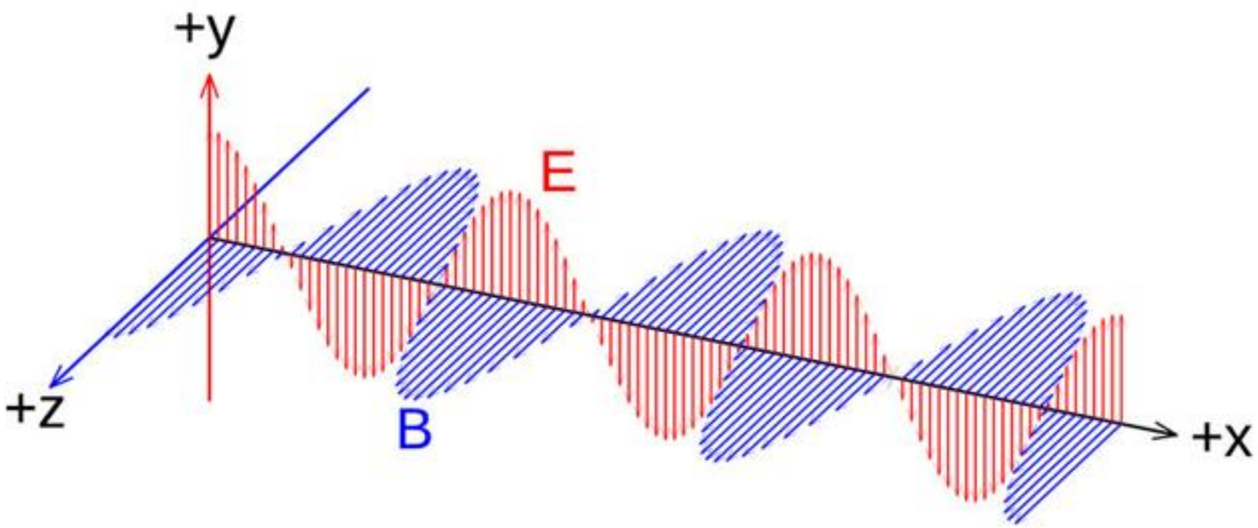


$\nabla \cdot \mathbf{D} = \rho$	(1)	Gauss' Law
$\nabla \cdot \mathbf{B} = 0$	(2)	Gauss' Law for magnetism
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	(3)	Faraday's Law
$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$	(4)	Ampère-Maxwell Law

Describe standing and propagating waves

https://en.wikipedia.org/wiki/Maxwell%27s_equations
https://www.powerelectronictips.com/intuitive-view-of-maxwells-equations-faq/

- Simplest solution of Maxwell's equations for wave propagation is for a **plane wave** in a dielectric material.
- **A plane wave** is a wave with a **constant amplitude** and **phase** in all directions perpendicular to its direction of travel.



$$E(x,t) = E_o \cos\left(\kappa' x - \omega t + \phi\right) ,$$

Similar equation for B

$\kappa' = 2\pi \,/\, \lambda$ is the angular wave number, and $\omega = 2\pi \,/\, T$ is the angular frequency;

https://www.oceanopticsbook.info/view/theory-electromagnetism/level-2/plane-wave-solutions#x1-1002r1

It is convenient to write the electric field as the real part of complex quantity.
Representing wave as complex quantity makes math easier.

Euler's Theorem: $e^{i\theta} = \cos\theta + i\sin\theta$

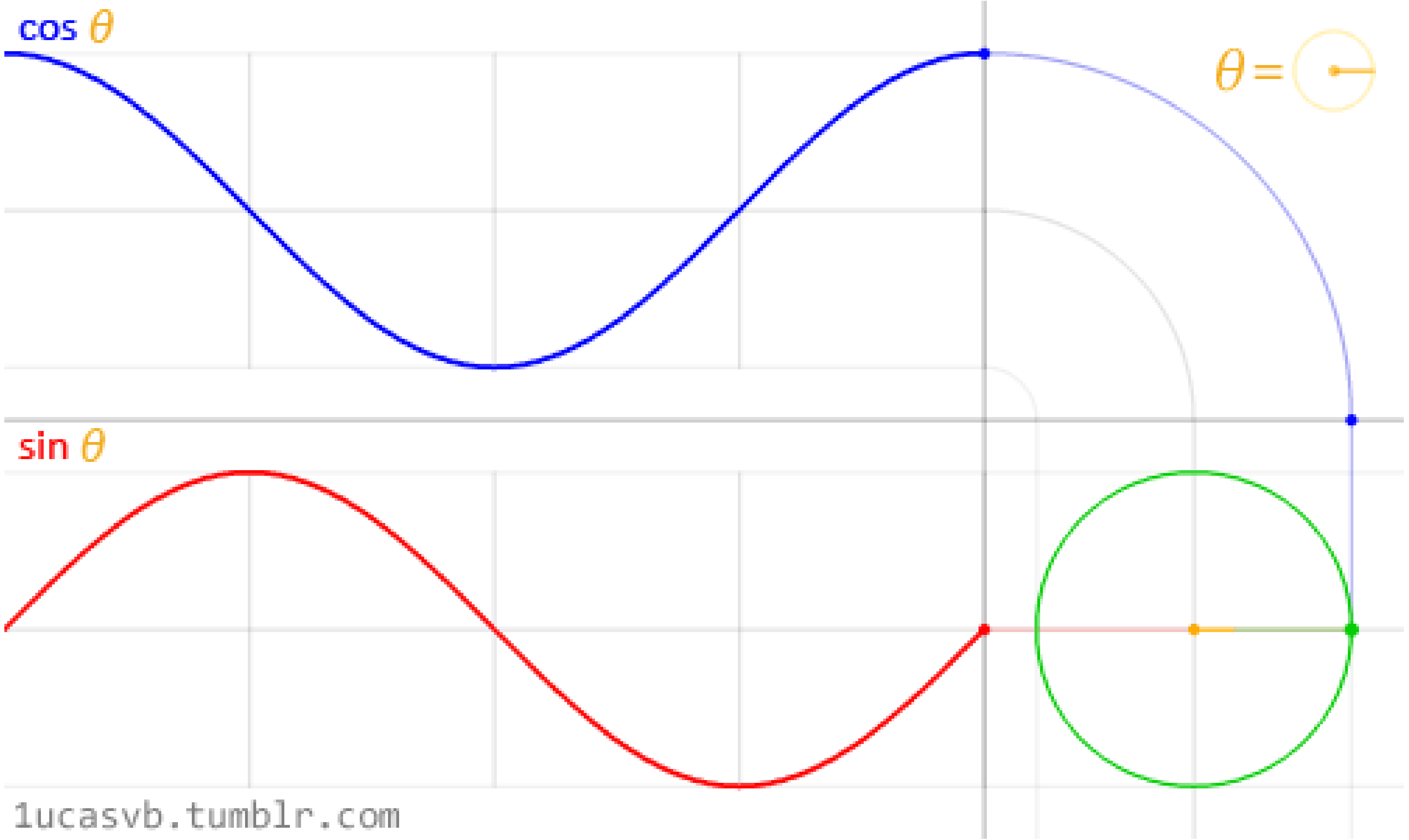
$$E(x,t) = E_o \cos\left(\kappa' x - \omega t + \phi\right),$$

$$E(x,t) = \Re\left\{E_o \exp i\left(\kappa' x - \omega t + \phi\right)\right\}$$

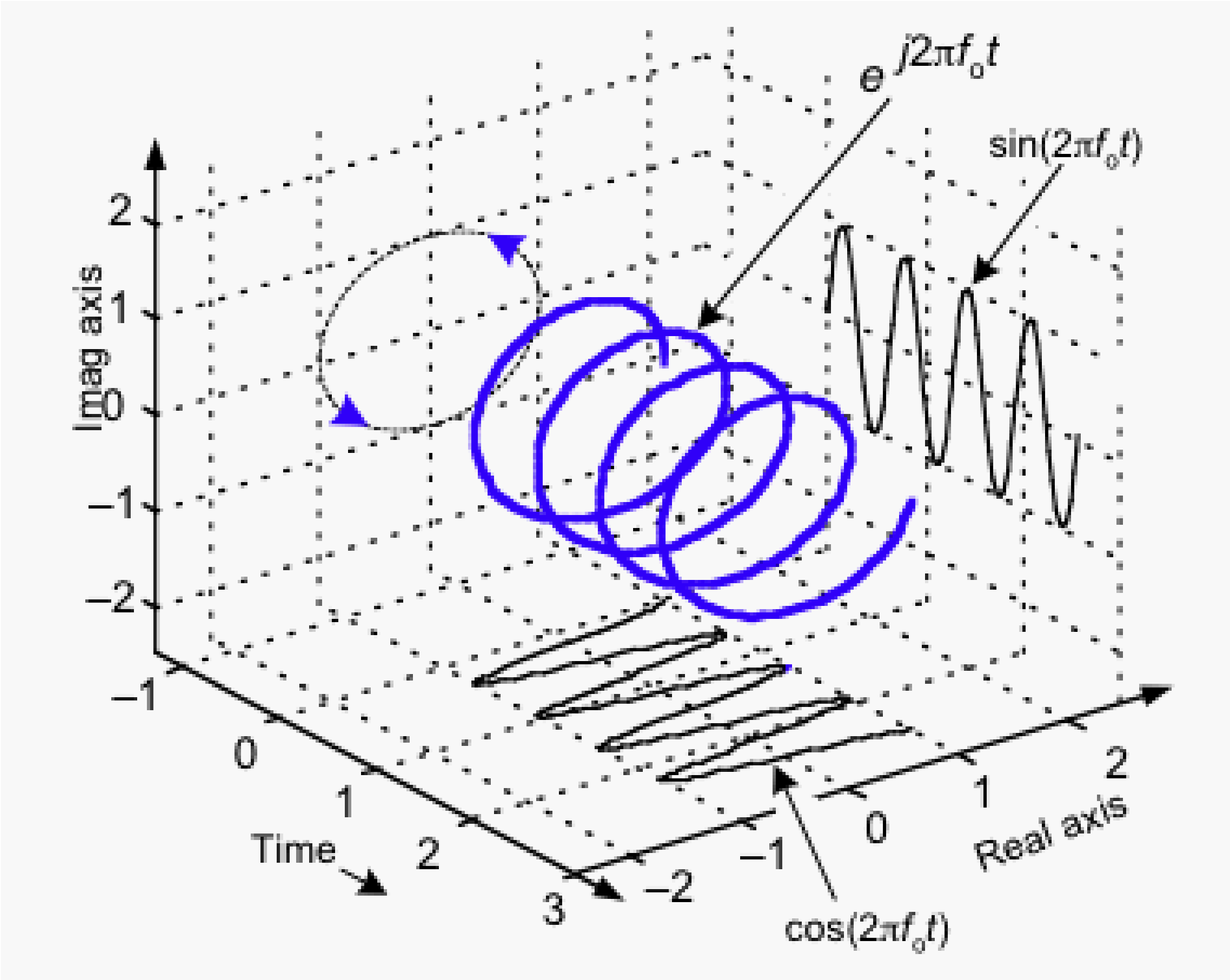
$$\widetilde{E}(x,t) = E_o \exp i\left(\kappa' x - \omega t + \phi\right)$$

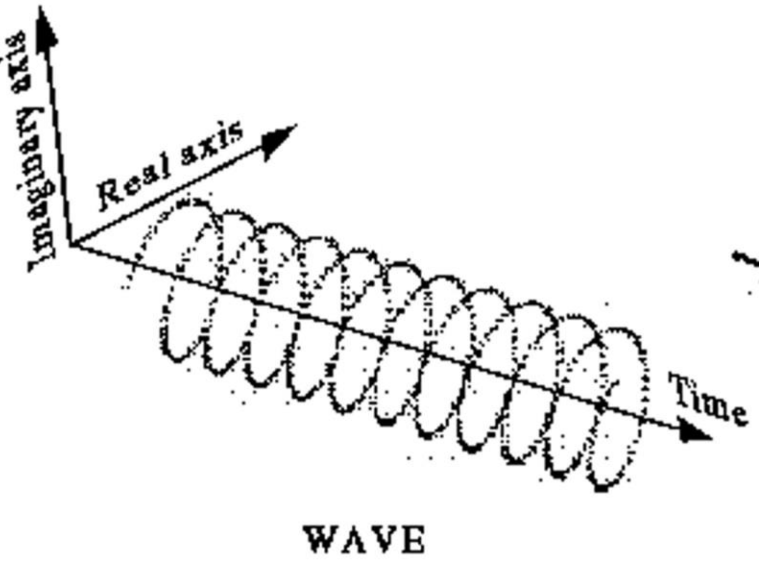
https://www.oceanopticsbook.info/view/theory-electromagnetism/level-2/plane-wave-solutions#x1-1002r1

https://phys.libretexts.org/Bookshelves/Quantum_Mechanics/Introductory_Quantum_Mechanics_(Fitzpatrick)/02%3A_Wave-Particle_Duality/2.03%3A_Representation_of_Waves_via_Complex_Functions

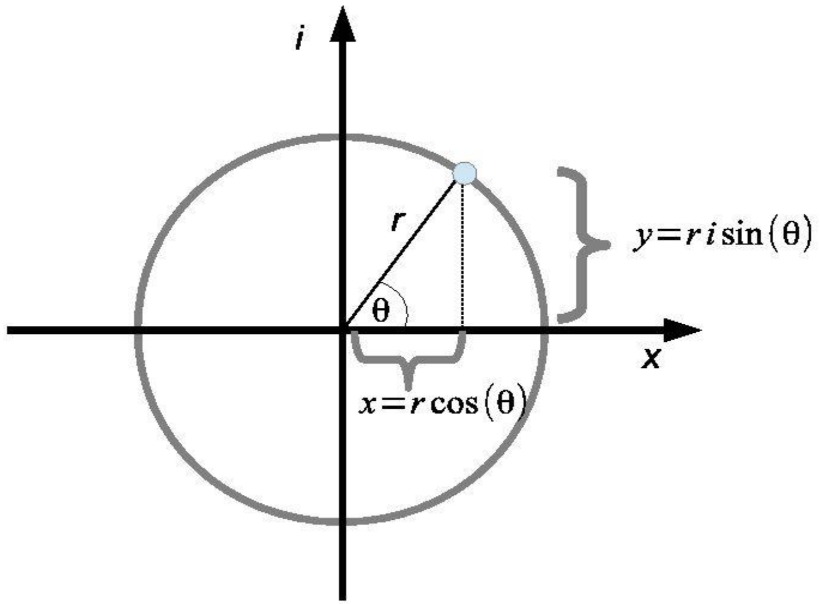


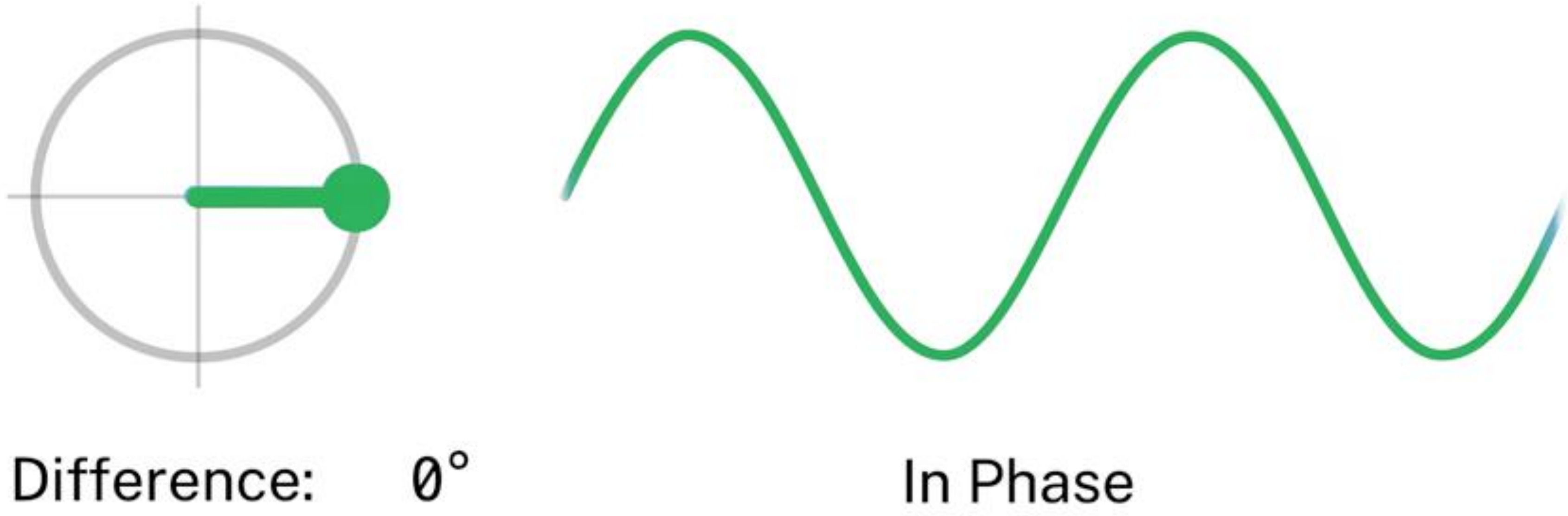
cos θ
sin θ
$\theta =$
1ucasvb.tumblr.com





https://math.stackexchange.com/questions/144268/is-there-a-name-for-this-type-of-plot-function-on-complex-plane-vs-time-shown



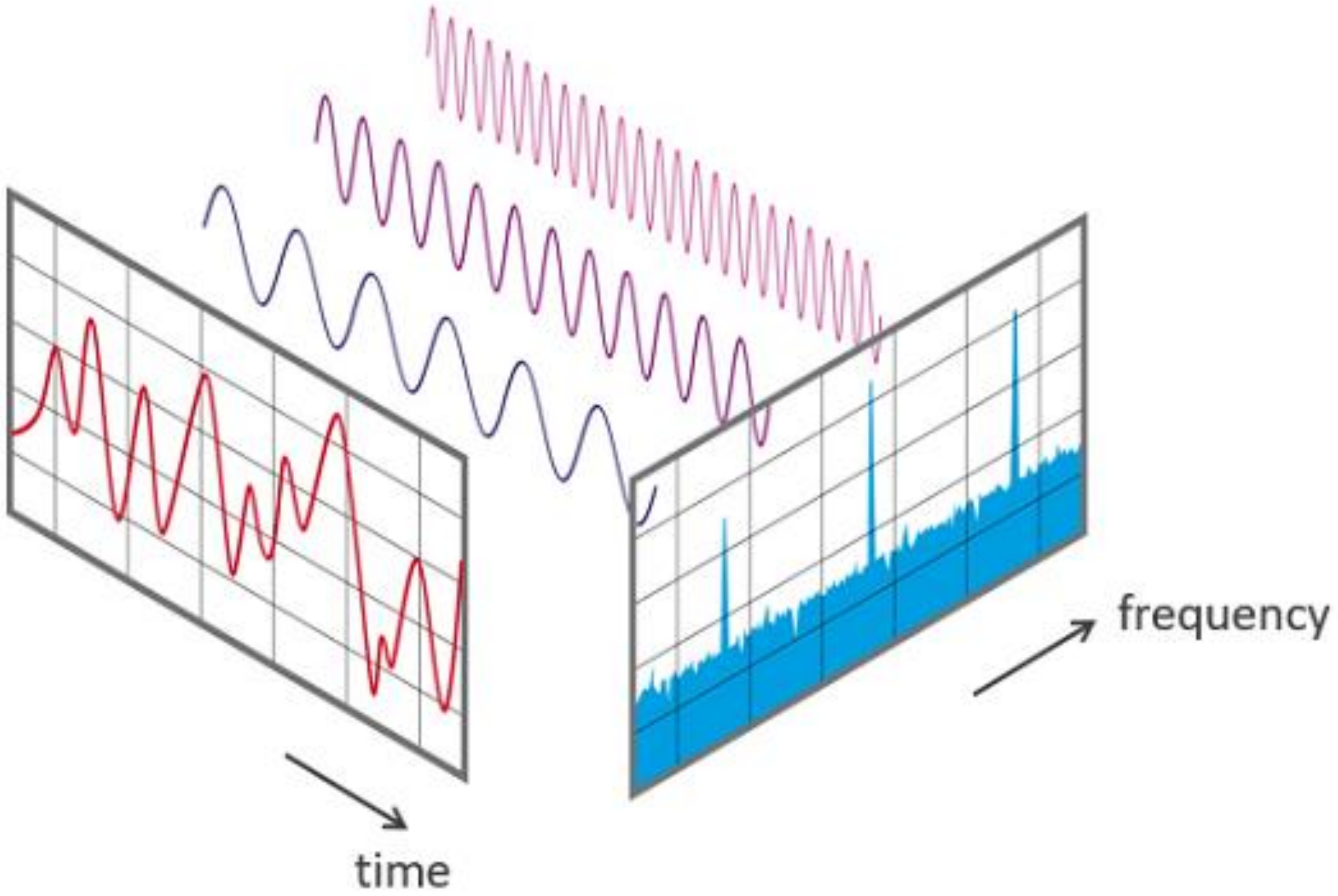


https://www.nist.gov/image/phase-differencegif

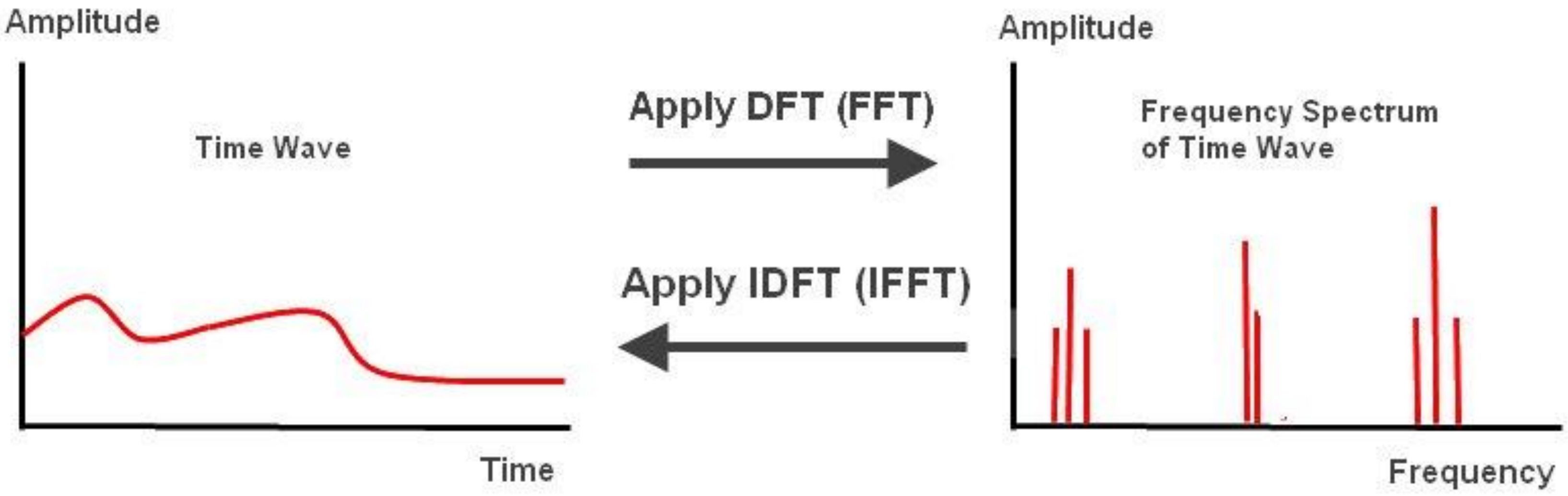
Fourier Transform & Discrete Fourier Transform (DFT)

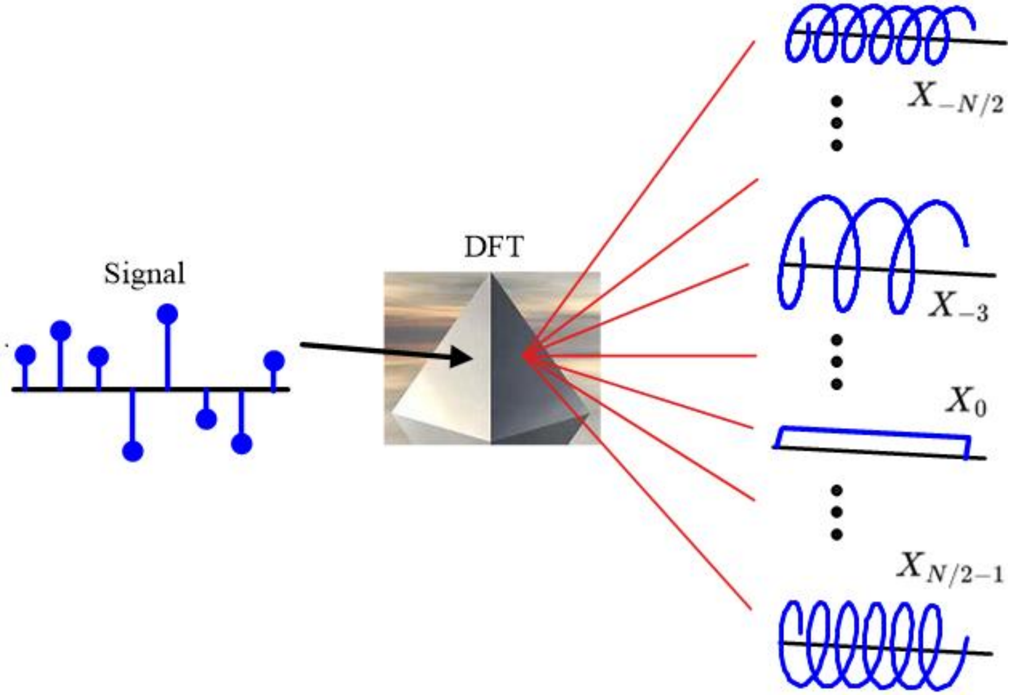
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi k n / N}$$



https://www.nti-audio.com/en/support/know-how/fast-fourier-transform-fft





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https://wirelesspi.com/the-discrete-fourier-transform-dft/

Sampling rate (f_s)

- **Meet the Nyquist theorem**: Your sampling frequency (f_s) must be at least twice the highest frequency (f_{max}) in your signal to avoid aliasing (Nyquist rate: $f_s \geq 2f_{max}$).
- **Practical considerations:** In practice, a higher sampling rate (e.g., 5 to 10 times the highest frequency) is often recommended for a more accurate representation and to ensure that your anti-aliasing filter can effectively remove frequencies above the Nyquist limit.

Number of samples (N)

- **Frequency resolution**: The frequency resolution of the DFT is determined by the sampling rate and the number of samples: $\Delta f = f_s / N$.

- **Symmetry:** For a real-valued input signal, its DFT will have a conjugate symmetry, meaning the second half of the spectrum is the complex conjugate of the first half. This property can be used to reduce computation.

- **Formula:** $f = k \cdot \frac{F_s}{N}$
  - f: Frequency in Hertz (Hz)
  - k: The index of the DFT bin (starting from 0)
  - F_s: The sample rate of the original signal in Hertz (Hz)
  - N: The total number of samples in the DFT (or FFT)
- **Example:** If your sampling rate (F_s) is 1000 Hz and you have performed an 8-point FFT ($N = 8$), the frequency for the bin with index $k = 2$ would be $f = 2 \cdot \frac{1000}{8} = 250$ Hz.

Discrete Fourier transform	$O\left(N^2\right)$
Fast Fourier transform	$O\left(Nlog_2N\right)$

Fast Fourier Transform (FFT)

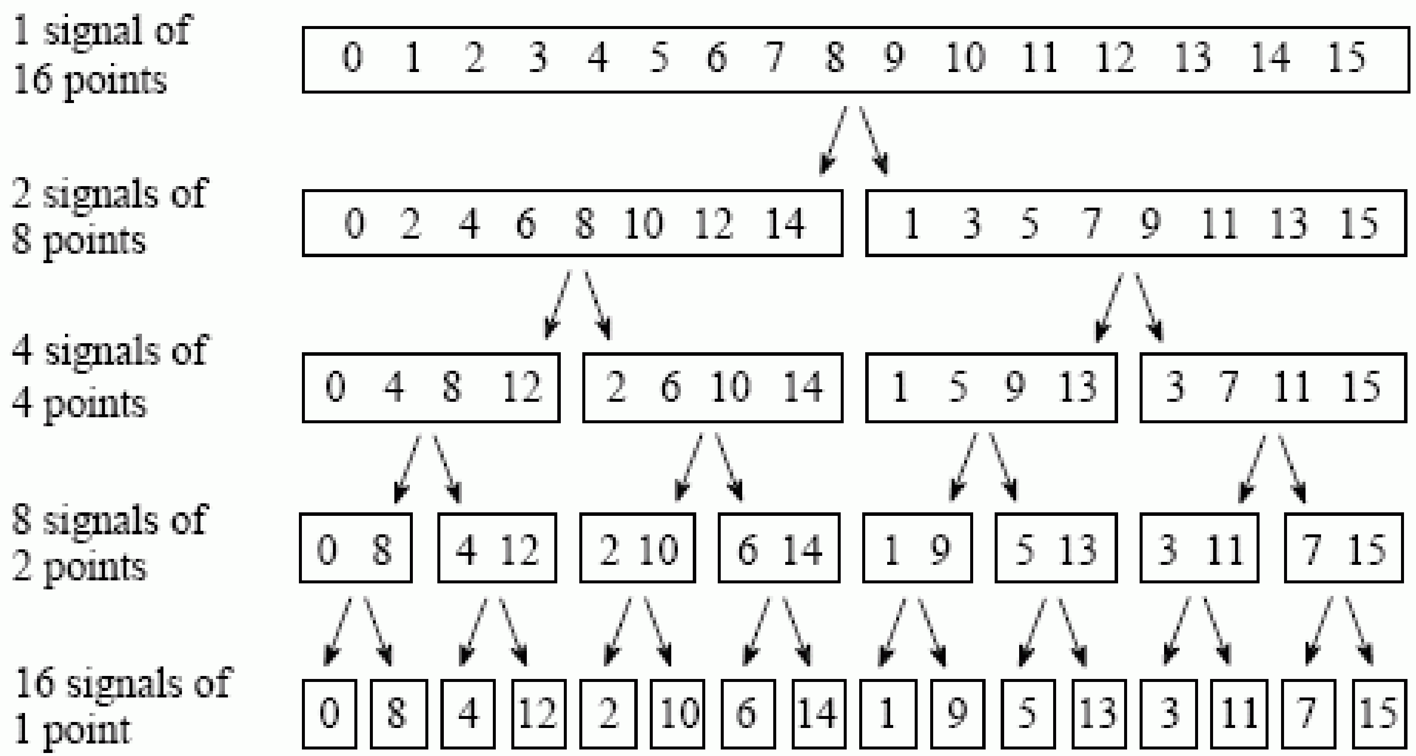

FIGURE 12-2
The FFT decomposition. An N point signal is decomposed into N signals each containing a single point. Each stage uses an *interlace decomposition*, separating the even and odd numbered samples.

https://www.dspguide.com/ch12/2.htm

$$\tilde{f}(k) = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} f(t)\,\omega_n^{-kt}\,. \qquad\qquad \omega_n = \mathrm{e}^{2\imath\pi/n}\,.$$

$f_{\text{even}}(s) = f(2s)$ and $f_{\text{odd}}(s) = f(2s+1)$, where s ranges from 0 to $n/2-1$.

$k = (n/2)k_0 + k'$, where $k_0 = 0$ or 1 and k' ranges from 0 to $n/2-1$.

$$\begin{aligned}
\tilde{f}(k) &= \frac{1}{\sqrt{n}} \left(\sum_{t \text{ even}} f(t)\,\omega_n^{-kt} + \sum_{t \text{ odd}} f(t)\,\omega_n^{-kt} \right) \\
&= \frac{1}{\sqrt{n}} \left(\sum_{s=0}^{n/2-1} f_{\text{even}}(s)\,\omega_n^{-2ks} + \omega_n^{-k} \sum_{s=0}^{n/2-1} f_{\text{odd}}(s)\,\omega_n^{-2ks} \right) \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{n/2}} \left(\sum_{s=0}^{n/2-1} f_{\text{even}}(s)\,\omega_{n/2}^{-k's} + (-1)^{k_0}\,\omega_n^{-k'} \sum_{s=0}^{n/2-1} f_{\text{odd}}(s)\,\omega_{n/2}^{-k's} \right) \\
&= \frac{1}{\sqrt{2}} \left(\tilde{f}_{\text{even}}(k') + (-1)^{k_0}\,\omega_n^{-k'}\,\tilde{f}_{\text{odd}}(k') \right).
\end{aligned}$$

$$\begin{aligned}
\omega_n^2 &= \omega_{n/2} \\
\omega_{n/2}^{-k} &= \mathrm{e}^{-2\imath\pi k_0}\,\omega_{n/2}^{-k'} = \omega_{n/2}^{-k'} \\
\omega_n^{-k} &= \mathrm{e}^{-\imath\pi k_0}\,\omega_{n/2}^{-k'} = (-1)^{k_0}\,\omega_{n/2}^{-k'}.
\end{aligned}$$

Cristopher Moore and Stephan Mertens. 2011. The Nature of Computation. Oxford University Press, Inc., USA.

HOMEWORK-3