Fast Fourier Transform

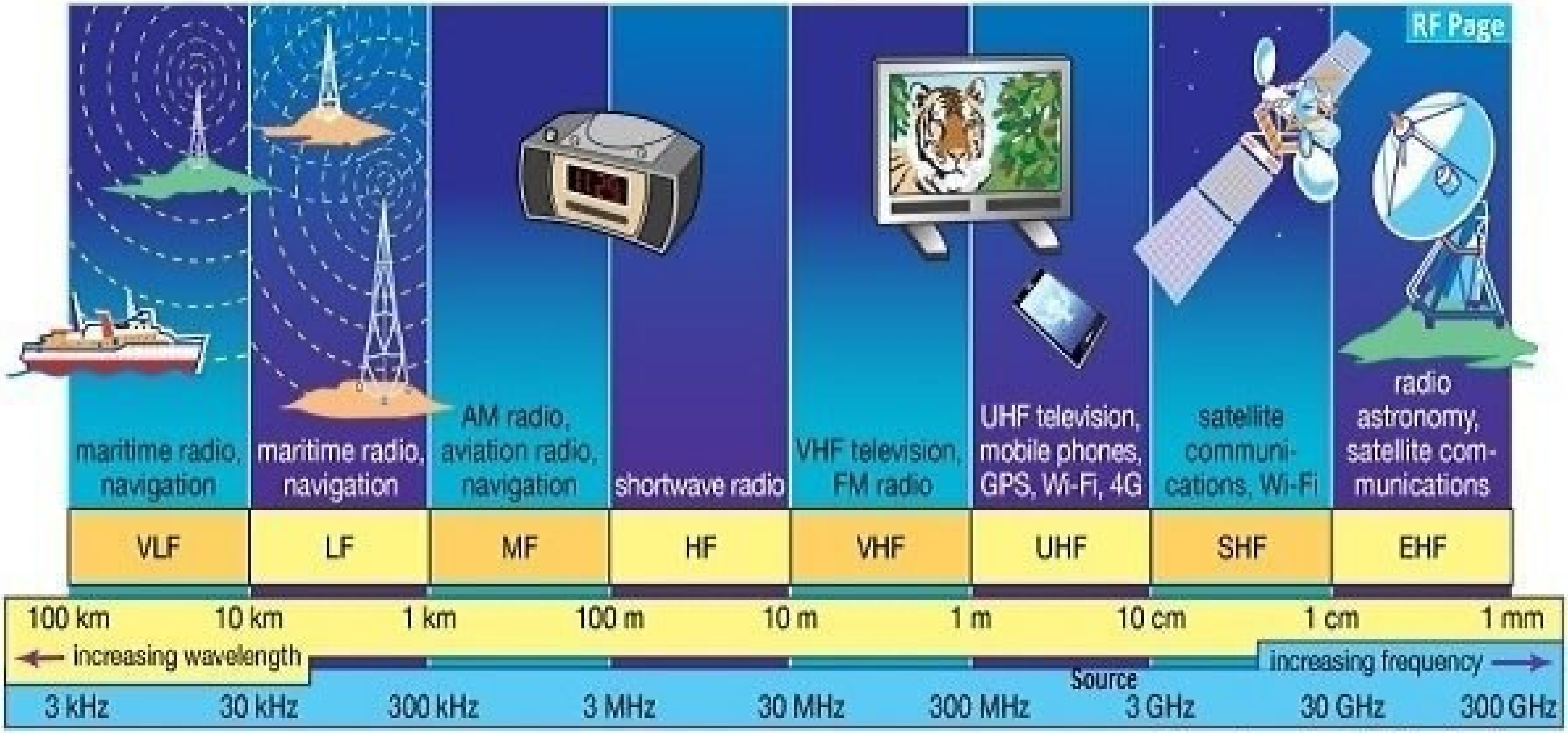
Slides by Siddharth S Ghule,
modified by jedimaestro@asu.edu with help from AI

CSE 468 Fall 2026

This next slide is about the first pop quiz, which is coming soon, and is not directly related to FFTs (the first exam will be about FFTs)...

Bitwise XOR as a cipher itself

- Typically used by malware, 8 or 32 bits
  - WEP attack uses these properties
- (B xor K) xor K = B
- (A xor K) xor (B xor K) = A xor B
- (0 xor K) = K
- (K xor K) = 0
- Frequency analysis or brute force



Source: Encyclopaedia Britannica, Inc.

https://www.rfpage.com/what-are-radio-frequency-bands-and-its-uses/#google_vignette

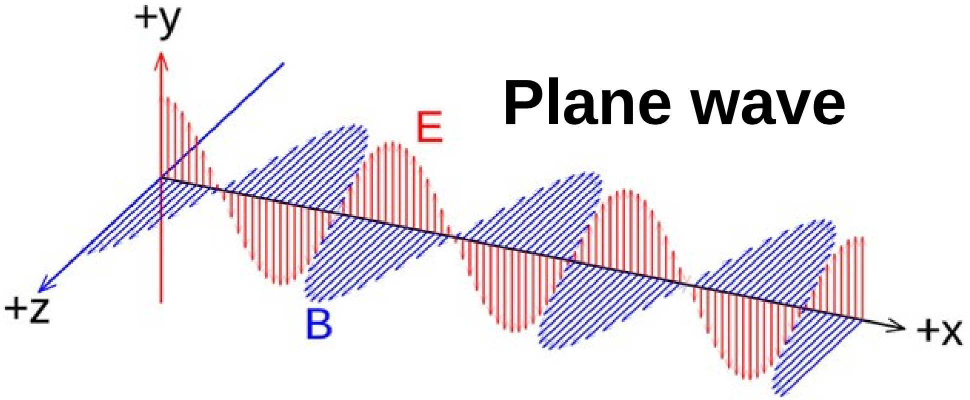
Representing wave as a complex quantity makes math easier.

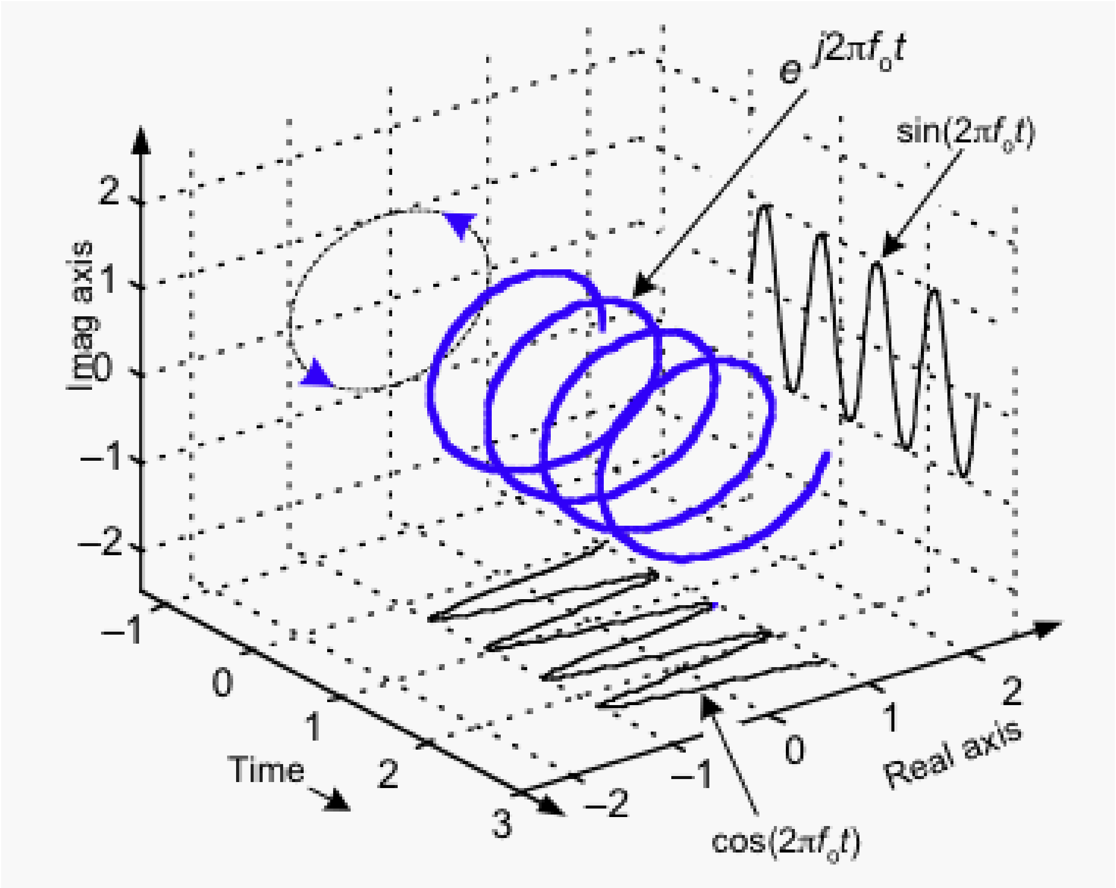
$$\widetilde{E}\,(x,t) = \quad E_{\text{o}} \exp i\,(\kappa' x - \omega t + \phi)$$

$$E\,(x,t) = \Re\left\{E_{\text{o}} \exp i\,(\kappa' x - \omega t + \phi)\right\}$$

Euler's Theorem: $e^{i\theta} = \cos\theta + i\sin\theta$

Plane wave





https://www.oceanopticsbook.info/view/theory-electromagnetism/level-2/plane-wave-solutions#x1-1002r1

https://math.stackexchange.com/questions/144268/is-there-a-name-for-this-type-of-plot-function-on-complex-plane-vs-time-shown

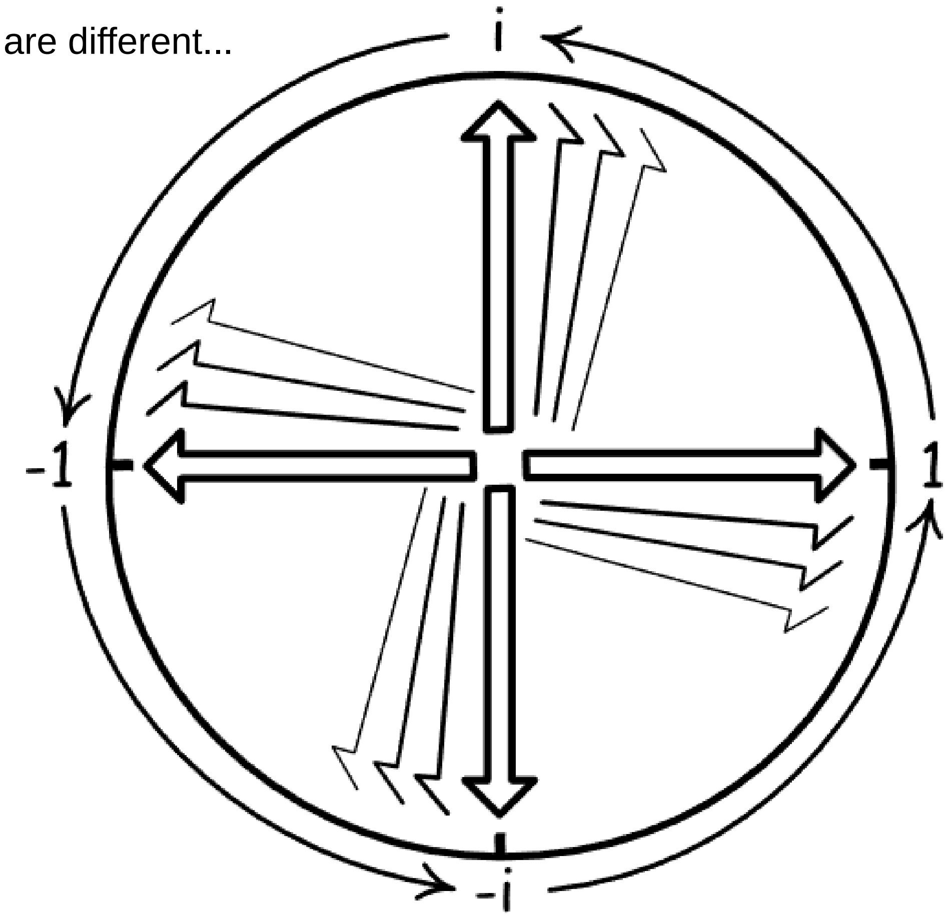
org/Bookshelves/Quantum_Mechanics/Introductory_Quantum_Mechanics_(Fitzpatrick)/02%3A_Wave-Particle_Duality/2.03%3A_
aves_via_Complex_Functions

$$\kappa = \frac{2\pi}{\lambda}$$ ← Wave number

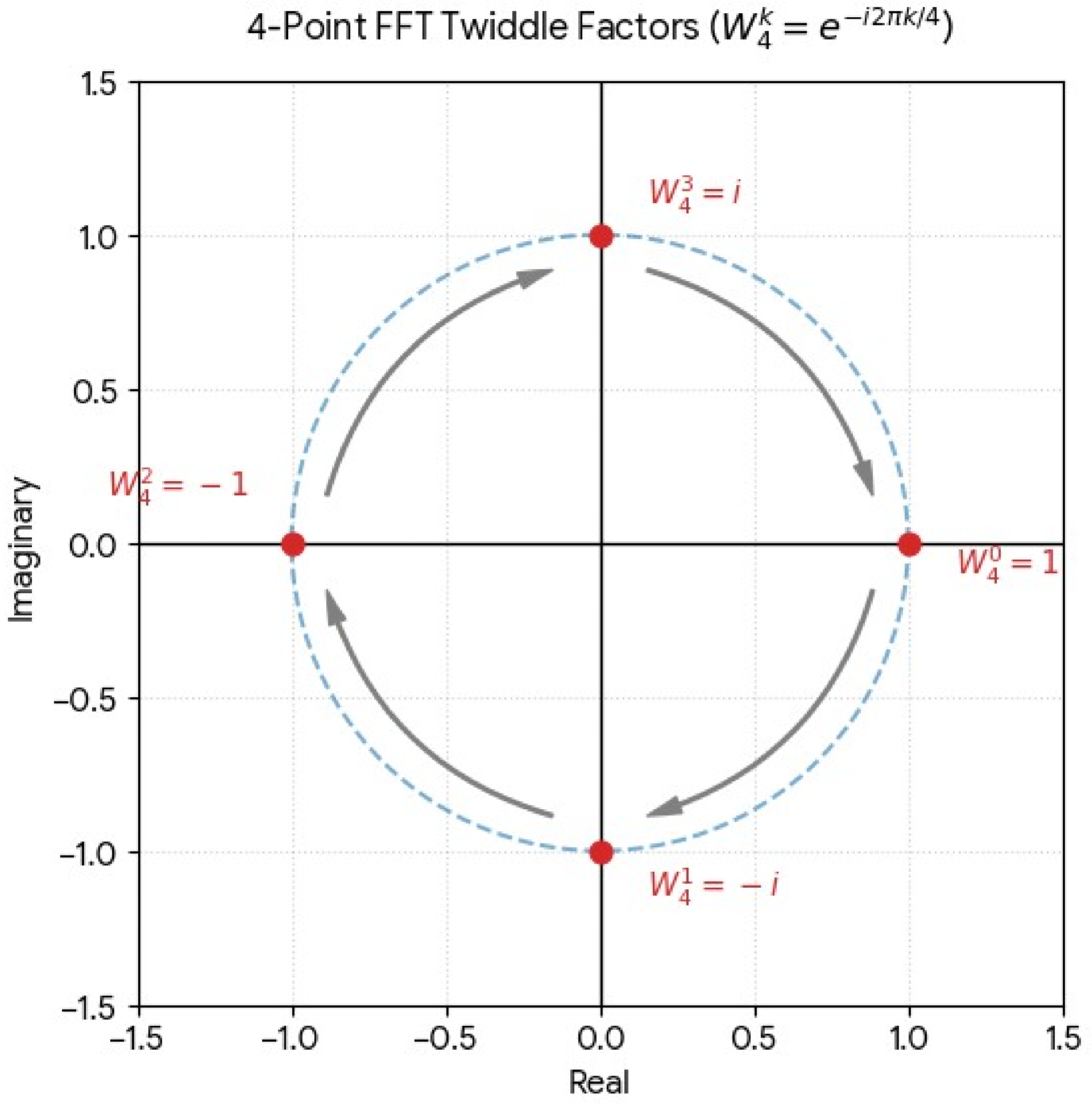
$$\omega = 2\pi\nu = \frac{2\pi}{T}$$ ← Angular frequency, ν is the frequency in Hz and T is the period

$$\phi$$ ← Phase shift

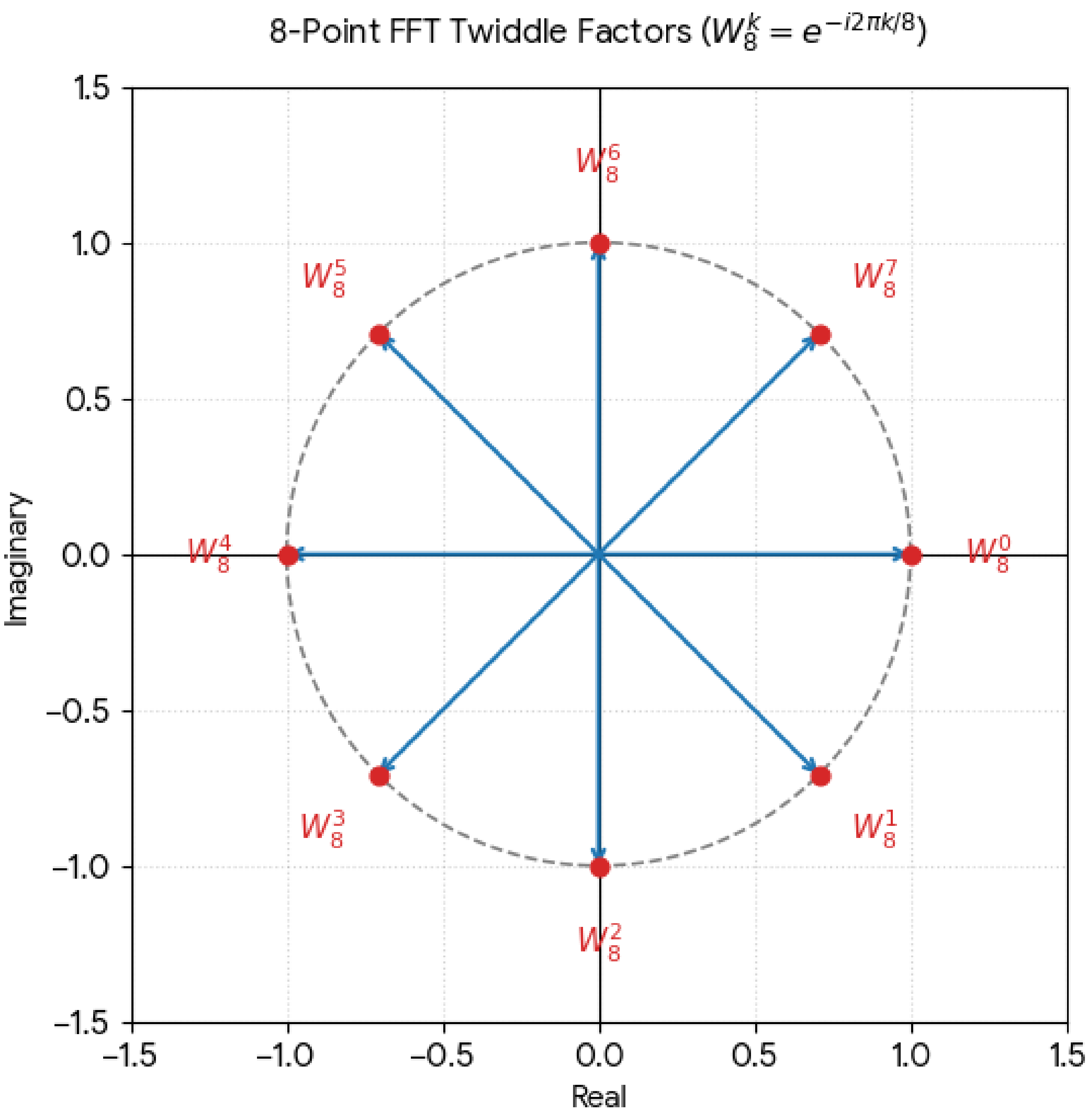
Complex waves are different...



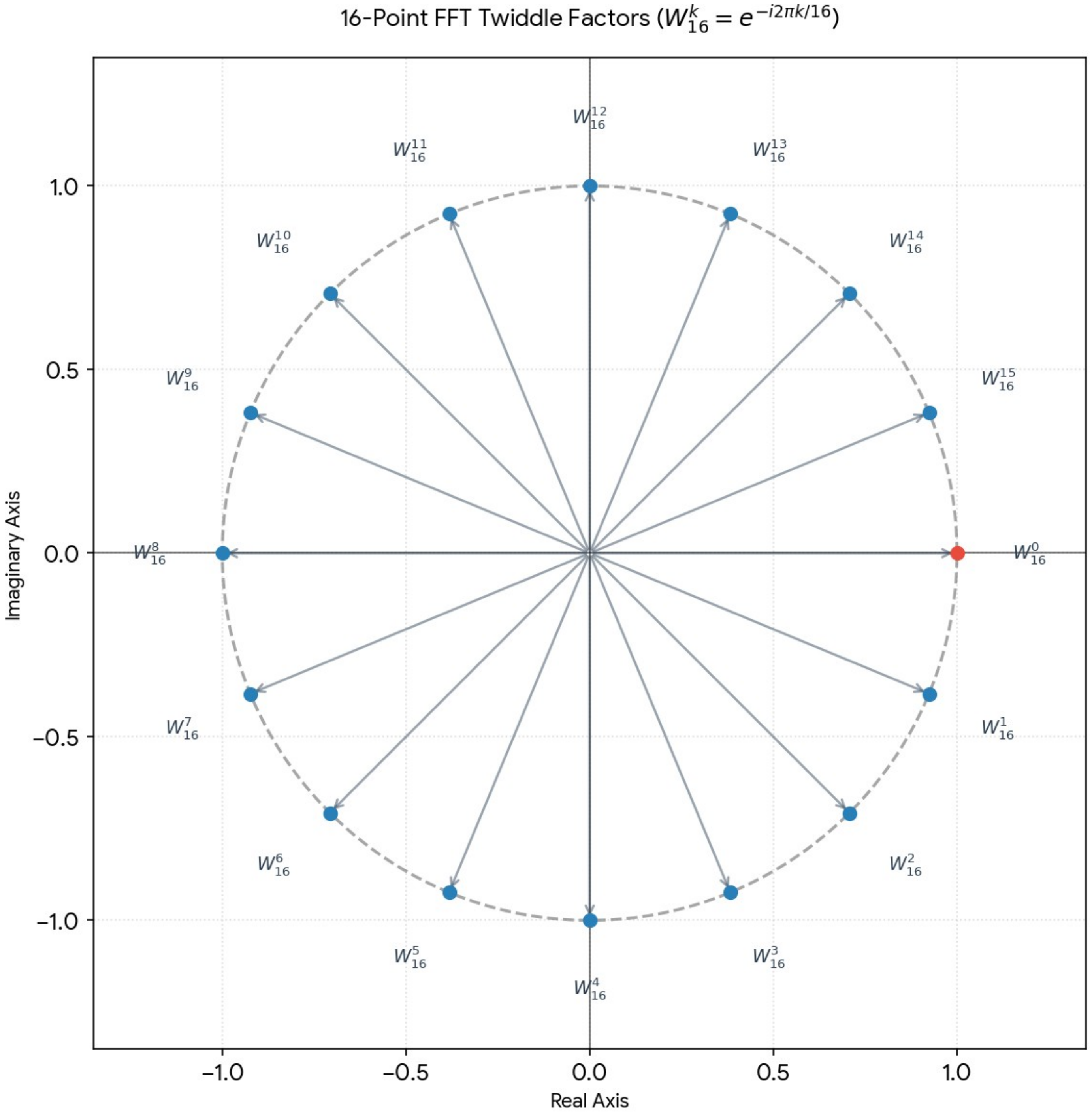
https://sites.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/quantum_theory_complex/complex.html



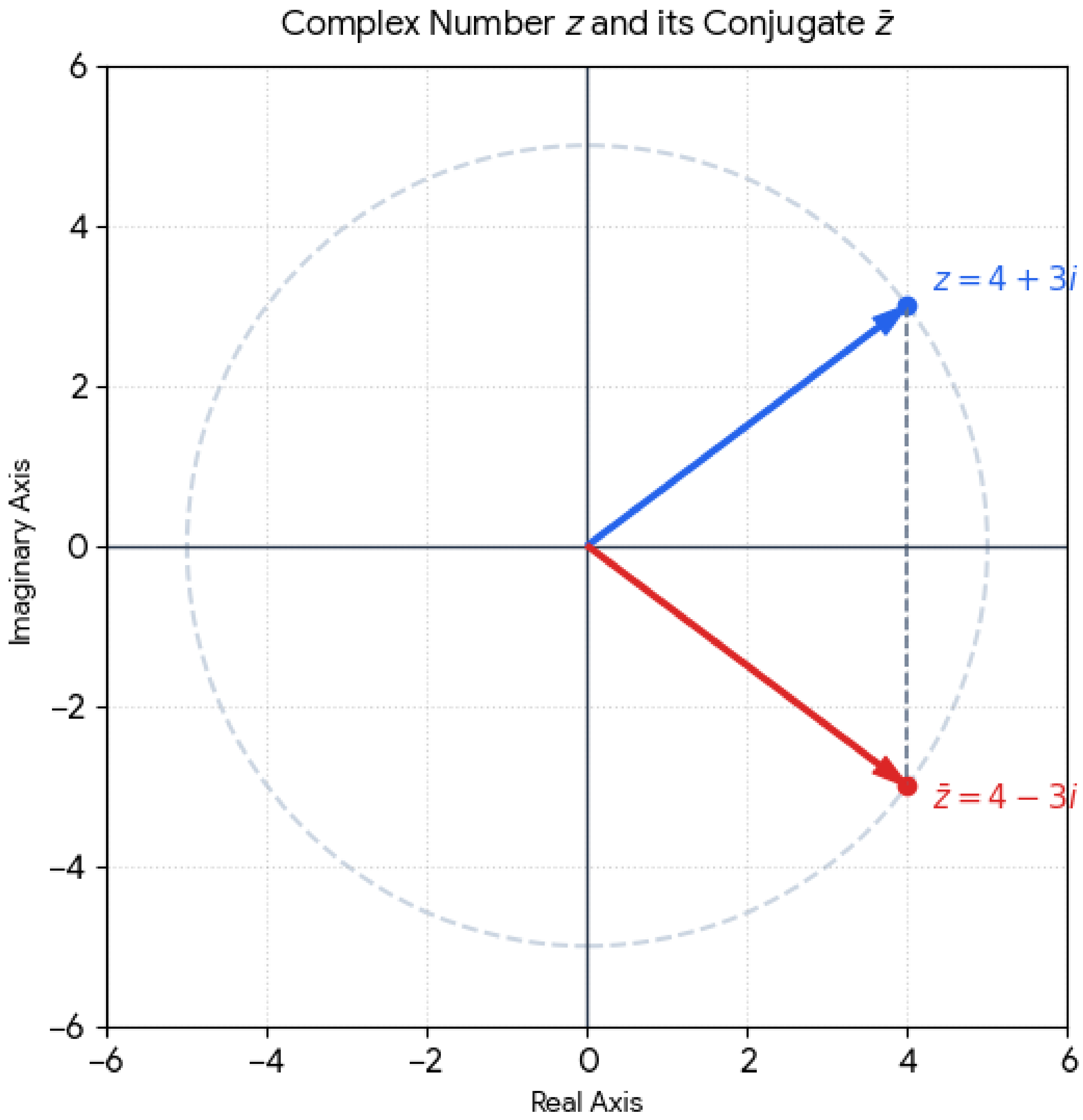
4-Point FFT Twiddle Factors ($W_4^k = e^{-i2\pi k/4}$)
$W_4^3 = i$
$W_4^2 = -1$
$W_4^0 = 1$
$W_4^1 = -i$
Imaginary
Real
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5



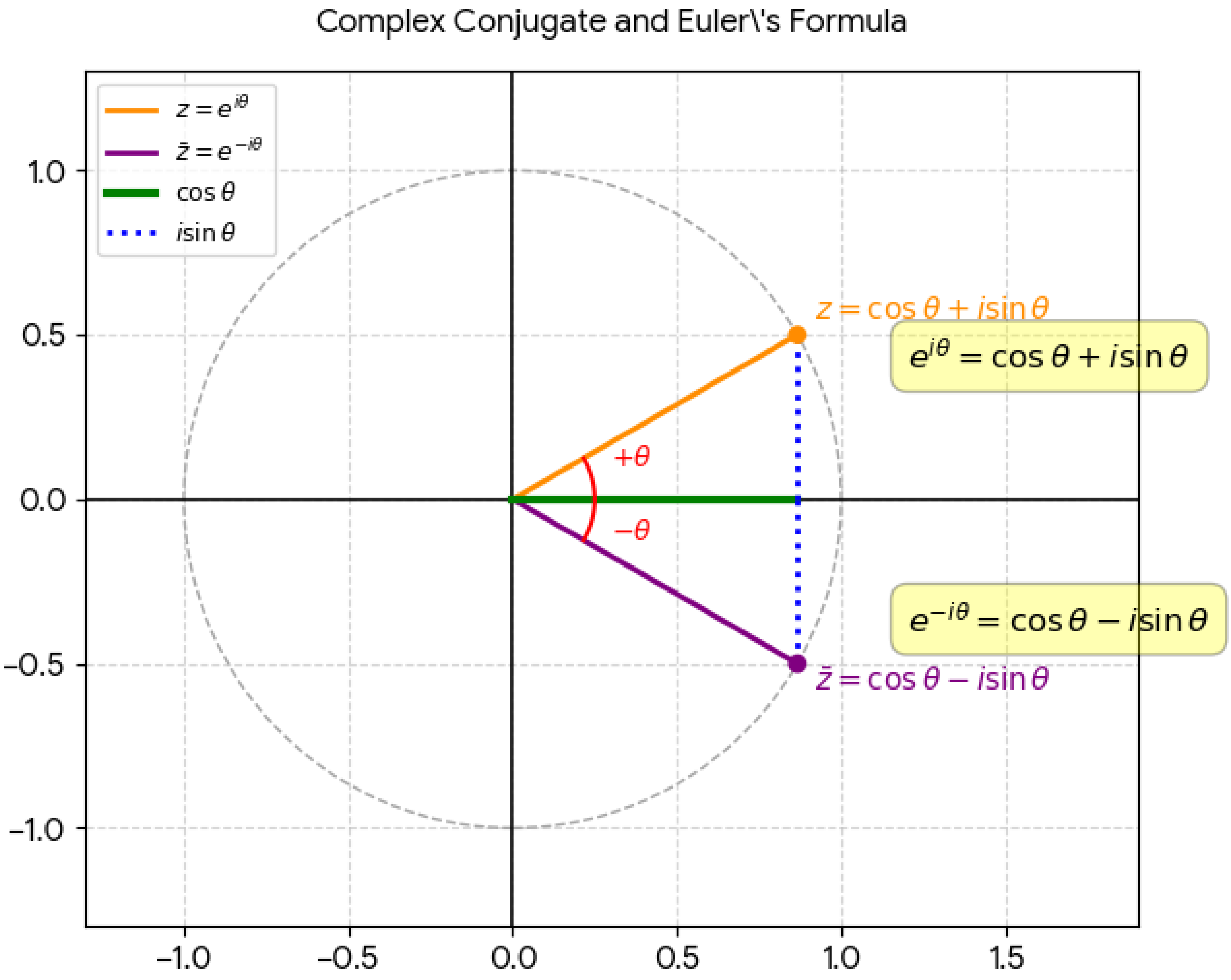
8-Point FFT Twiddle Factors ($W_8^k = e^{-i2\pi k/8}$)
W_8^0
W_8^1
W_8^2
W_8^3
W_8^4
W_8^5
W_8^6
W_8^7
Imaginary
Real
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5



16-Point FFT Twiddle Factors ($W_{16}^{k} = e^{-i2\pi k/16}$)
Imaginary Axis
Real Axis
1.0
0.5
0.0
−0.5
−1.0
W_{16}^{0}
W_{16}^{1}
W_{16}^{2}
W_{16}^{3}
W_{16}^{4}
W_{16}^{5}
W_{16}^{6}
W_{16}^{7}
W_{16}^{8}
W_{16}^{9}
W_{16}^{10}
W_{16}^{11}
W_{16}^{12}
W_{16}^{13}
W_{16}^{14}
W_{16}^{15}



Complex Number z and its Conjugate $\bar{z}$
$z = 4 + 3i$
$\bar{z} = 4 - 3i$
Imaginary Axis
Real Axis
6
4
2
0
−2
−4
−6


Complex Conjugate and Euler\'s Formula
$z = e^{i\theta}$
$\bar{z} = e^{-i\theta}$
$\cos\theta$
$i\sin\theta$
$z = \cos\theta + i\sin\theta$
$e^{i\theta} = \cos\theta + i\sin\theta$
$+\theta$
$-\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$
$\bar{z} = \cos\theta - i\sin\theta$
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
1.5

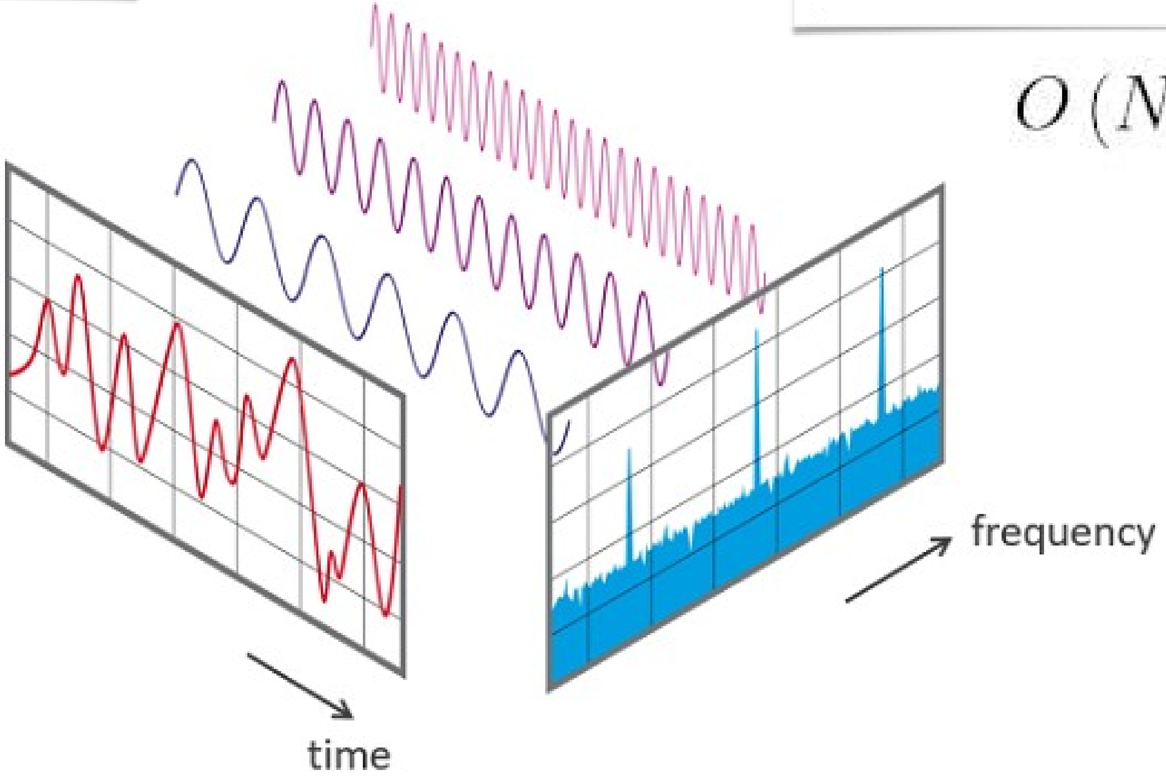
Fourier Transform & Discrete Fourier Transform (DFT)

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$O\left(N^2\right)$$

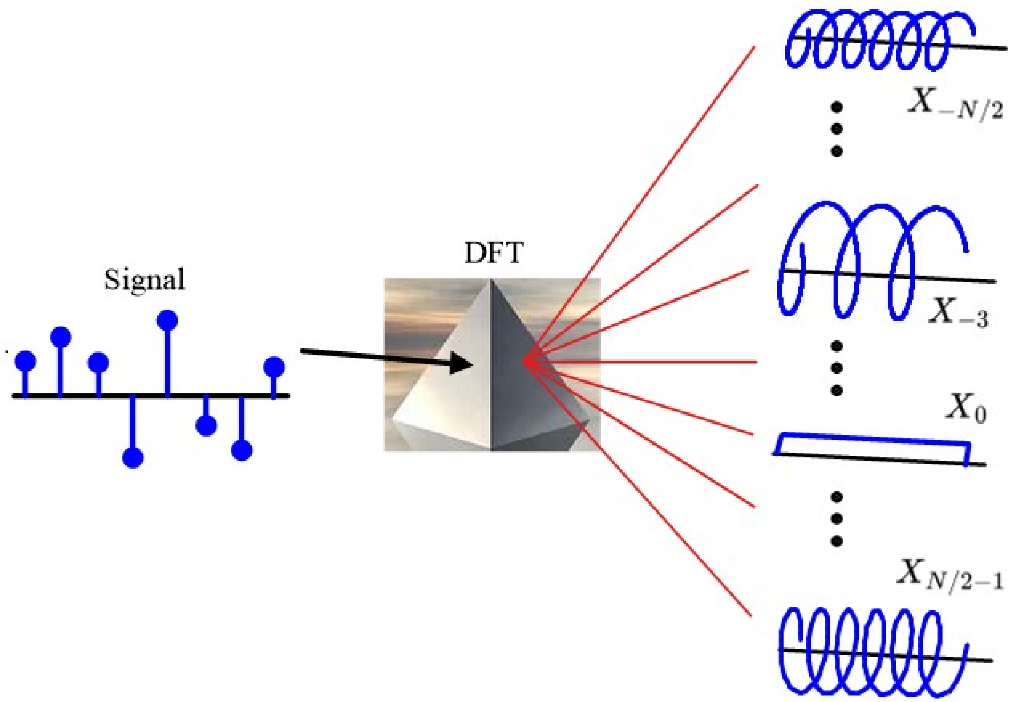
$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

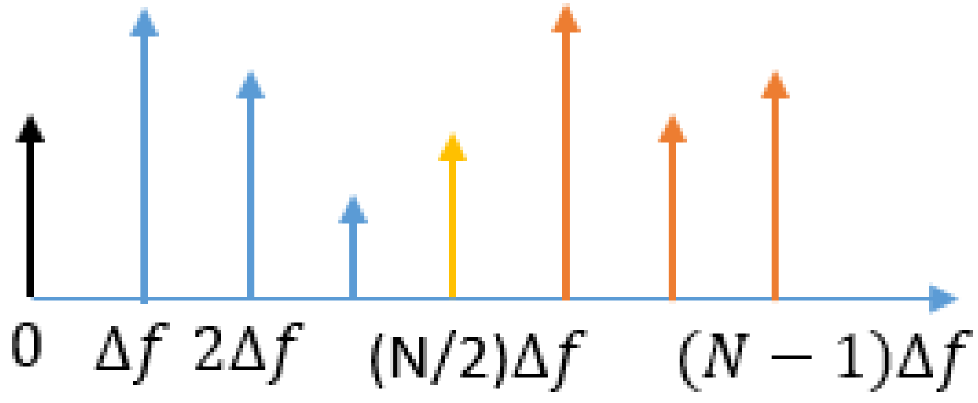
$$O\left(N log_2 N\right)$$



https://www.nti-audio.com/en/support/know-how/fast-fourier-transform-fft

DFT





$$\Delta f = \frac{F_s}{N}$$

F_s = Sampling frequency

N = Total number of input samples

Frequency Range = $0, \ldots, F_s$

https://wirelesspi.com/the-discrete-fourier-transform-dft/

https://www.gaussianwaves.com/2015/11/interpreting-fft-results-complex-dft-frequency-bins-and-fftshift/

$$f_s > 2 \cdot f_{max}$$

Humans can hear frequencies up to 20 kHz

Spotify *etc.* sample music at 44.1 kHz

FFT

$$\underbrace{X[k]}_{\text{Frequency bins}} = \sum_{n=0}^{\overbrace{N}^{\text{Total Number of Samples}}-1} \underbrace{x[n]}_{\text{Input Samples}} \underbrace{W_N^{nk}}_{\text{Twiddle Factor}}$$

$$X[k] = E[k] + \omega_k \cdot O[k]$$

$$X[k + \tfrac{N}{2}] = E[k] - \omega_k \cdot O[k]$$

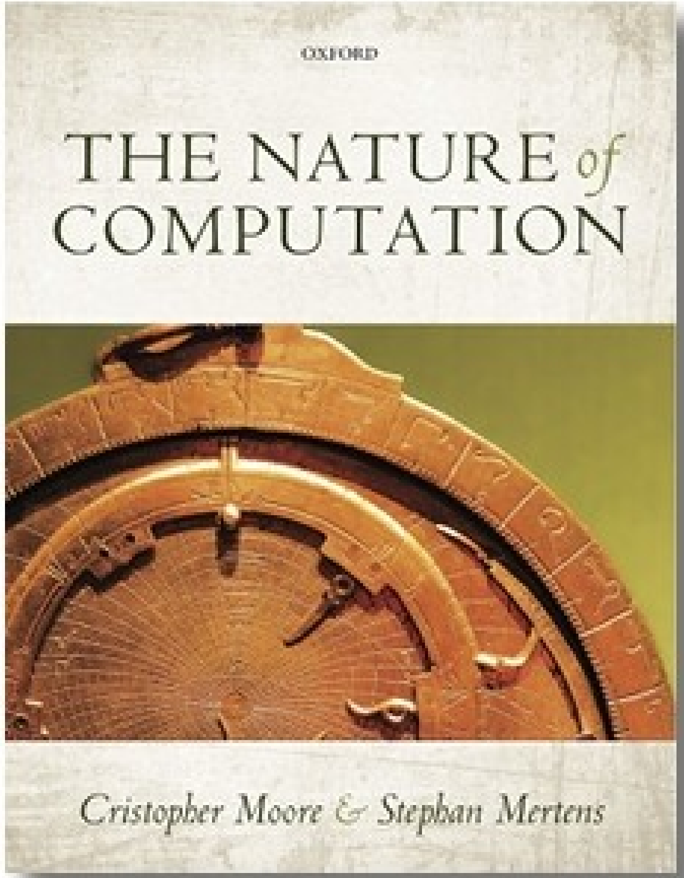
$$k = 0 \text{ to } \frac{N}{2} - 1$$

https://www.hesliplabs.com/blog/writing-your-own-fft-for-python-amp-stm32

Cristopher Moore and Stephan Mertens. 2011. The Nature of Computation. Oxford University Press, Inc., USA.

$$\begin{pmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \end{pmatrix} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \\ 1 & \omega_n & \omega_n^2 & \cdots \\ 1 & \omega_n^2 & \omega_n^4 & \\ & \vdots & & \ddots \end{pmatrix} \cdot \begin{pmatrix} \tilde{f}(0) \\ \tilde{f}(1) \\ \tilde{f}(2) \\ \vdots \end{pmatrix}$$

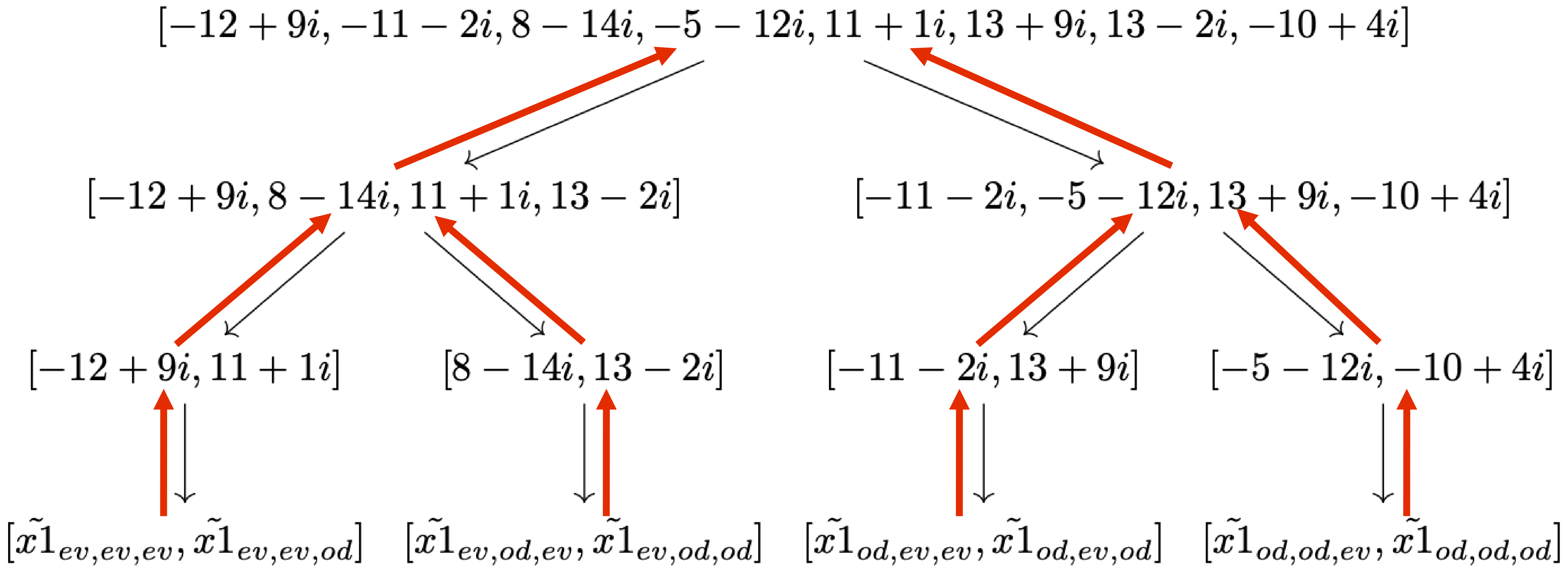
$$f = Q \cdot \tilde{f} \text{ where } Q_{tk} = \frac{1}{\sqrt{n}} \omega_n^{kt} .$$



https://nature-of-computation.org/

$$[-12+9i, -11-2i, 8-14i, -5-12i, 11+1i, 13+9i, 13-2i, -10+4i]$$

$$
\begin{tikzcd}[column sep=tiny]
 & & & {[-12+9i, -11-2i, 8-14i, -5-12i, 11+1i, 13+9i, 13-2i, -10+4i]} \arrow[dll] \arrow[drr] & & & \\
 & {[-12+9i, 8-14i, 11+1i, 13-2i]} \arrow[dl] \arrow[dr] & & & & {[-11-2i, -5-12i, 13+9i, -10+4i]} \arrow[dl] \arrow[dr] & \\
{[-12+9i, 11+1i]} \arrow[d] & & {[8-14i, 13-2i]} \arrow[d] & & {[-11-2i, 13+9i]} \arrow[d] & & {[-5-12i, -10+4i]} \arrow[d] \\
{[\tilde{x1}_{ev,ev,ev}, \tilde{x1}_{ev,ev,od}]} & & {[\tilde{x1}_{ev,od,ev}, \tilde{x1}_{ev,od,od}]} & & {[\tilde{x1}_{od,ev,ev}, \tilde{x1}_{od,ev,od}]} & & {[\tilde{x1}_{od,od,ev}, \tilde{x1}_{od,od,od}]}
\end{tikzcd}
$$


$[-12+9i, -11-2i, 8-14i, -5-12i, 11+1i, 13+9i, 13-2i, -10+4i]$
$[-12+9i, 8-14i, 11+1i, 13-2i]$
$[-11-2i, -5-12i, 13+9i, -10+4i]$
$[-12+9i, 11+1i]$
$[8-14i, 13-2i]$
$[-11-2i, 13+9i]$
$[-5-12i, -10+4i]$
$[\tilde{x1}_{ev,ev,ev}, \tilde{x1}_{ev,ev,od}]$
$[\tilde{x1}_{ev,od,ev}, \tilde{x1}_{ev,od,od}]$
$[\tilde{x1}_{od,ev,ev}, \tilde{x1}_{od,ev,od}]$
$[\tilde{x1}_{od,od,ev}, \tilde{x1}_{od,od,od}]$

Twiddle Factors

$$\omega_k = e^{-2\pi i k/2}$$

$$\omega_0 : \quad e^{-2\pi i \cdot 0/8} = e^0 = 1$$

$$\omega_1 : \quad e^{-2\pi i \cdot 1/8} = e^{-\pi i/4} = \cos(\tfrac{-\pi}{4}) + i\sin(\tfrac{-\pi}{4}) = \tfrac{1-i}{\sqrt{2}}$$

$$\omega_2 : \quad e^{-2\pi i \cdot 2/8} = e^{-\pi i/2} = \cos(\tfrac{-\pi}{2}) + i\sin(\tfrac{-\pi}{2}) = -i$$

$$\omega_3 : \quad e^{-2\pi i \cdot 3/8} = e^{-\pi i 3/4} = \cos(\tfrac{-3\pi}{4}) + i\sin(\tfrac{-3\pi}{4}) = -\tfrac{1+i}{\sqrt{2}}$$

What is the 1-point FFT of each of these samples?

- x = [4]
- x = [-16]
- x = [7 + 5i]
- x = [9 – 2i]

What is the 1-point FFT of each of these samples?

- x = [4] → X = [4]
- x = [-16] → X = [-16]
- x = [7 + 5i] → X = [7 + 5i]
- x = [9 – 2i] → X = [9 – 2i]

2-point FFT of x = [7, 16]

$$W_2^0 = e^{-i2\pi(0)/2} = 1$$

$$X(0) = x(0) + W_2^0 \cdot x(1)$$

$$X(1) = x(0) - W_2^0 \cdot x(1)$$

$$X(0) = 7 + 1 \cdot 16 = 7 + 16 = 23$$

$$X(1) = 7 - 1 \cdot 16 = 7 - 16 = -9$$

Answer is X = [23, -9]

$$x = [7 + 6i, 3 - 2i, 4 - 3i, \ -5 - i]$$

$$E(0) = x(0) + x(2) = (7 + 6i) + (4 - 3i) = 11 + 3i$$

$$E(1) = x(0) - x(2) = (7 + 6i) - (4 - 3i) = 3 + 9i$$

$$O(0) = x(1) + x(3) = (3 - 2i) + (-5 - i) = -2 - 3i$$

$$O(1) = x(1) - x(3) = (3 - 2i) - (-5 - i) = 8 - i$$

$W_4^0 = 1$ and $W_4^1 = -i$.

$$X(0) = E(0) + W_4^0 \cdot O(0) = (11 + 3i) + 1 \cdot (-2 - 3i) = 9$$

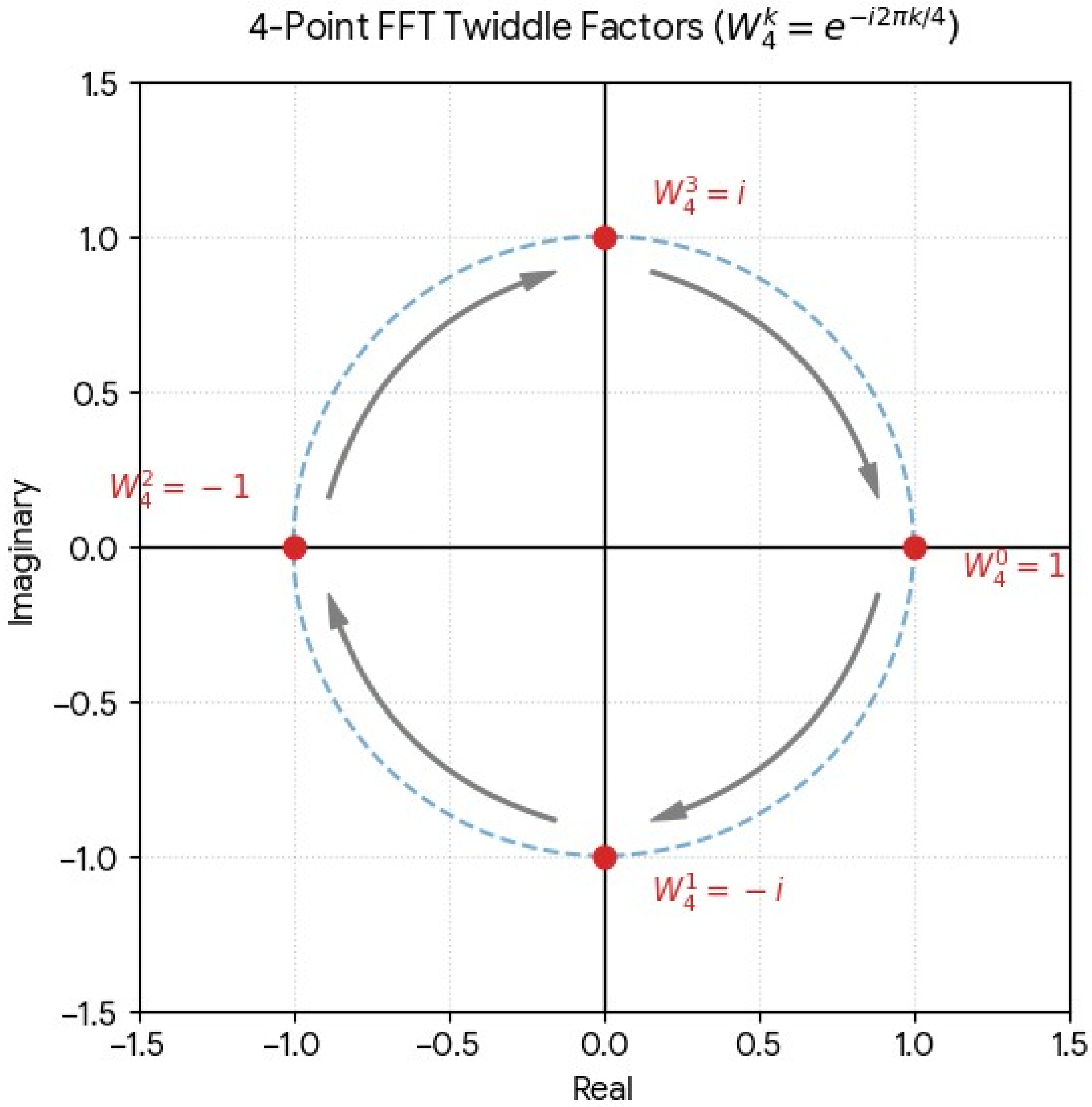
$$X(1) = E(1) + W_4^1 \cdot O(1) = (3 + 9i) + (-i) \cdot (8 - i) = 2 + i$$

$$X(2) = E(0) - W_4^0 \cdot O(0) = (11 + 3i) - 1 \cdot (-2 - 3i) = 13 + 6i$$

$$X(3) = E(1) - W_4^1 \cdot O(1) = (3 + 9i) - (-i) \cdot (8 - i) = 4 + 17i$$

$$x = [7 + 6i, 3 - 2i, 4 - 3i, -5 - i]$$

$$X = [9, 2 + i, 13 + 6i, 4 + 17i]$$



4-Point FFT Twiddle Factors ($W_4^k = e^{-i2\pi k/4}$)
$W_4^3 = i$
$W_4^2 = -1$
$W_4^0 = 1$
$W_4^1 = -i$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
Imaginary
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
Real

$$x = [1, 8, -3, 2.9]$$

$$E(0) = x(0) + x(2) = 1 + (-3) = -2$$

$$E(1) = x(0) - x(2) = 1 - (-3) = 4$$

$$O(0) = x(1) + x(3) = 8 + 2.9 = 10.9$$

$$O(1) = x(1) - x(3) = 8 - 2.9 = 5.1$$

$W_4^0 = 1$ and $W_4^1 = -i.$

$$X(0) = E(0) + W_4^0 \cdot O(0) = -2 + 1 \cdot 10.9 = 8.9$$

$$X(1) = E(1) + W_4^1 \cdot O(1) = 4 + (-i) \cdot 5.1 = 4 - 5.1i$$

$$X(2) = E(0) - W_4^0 \cdot O(0) = -2 - 1 \cdot 10.9 = -12.9$$

$$X(3) = E(1) - W_4^1 \cdot O(1) = 4 - (-i) \cdot 5.1 = 4 + 5.1i$$

$$x = [1, 8, -3, 2.9]$$

$$X = [8.9, 4 - 5.1i, -12.9, 4 + 5.1i]$$

DC component

Nyquist frequency

Complex conjugates

soundmeterx.com/sound-spectrum-analyzer/

Sound Spectrum Analyzer

peak, RMS, crest factor, L/R correlation, dominant frequency, and band energy.

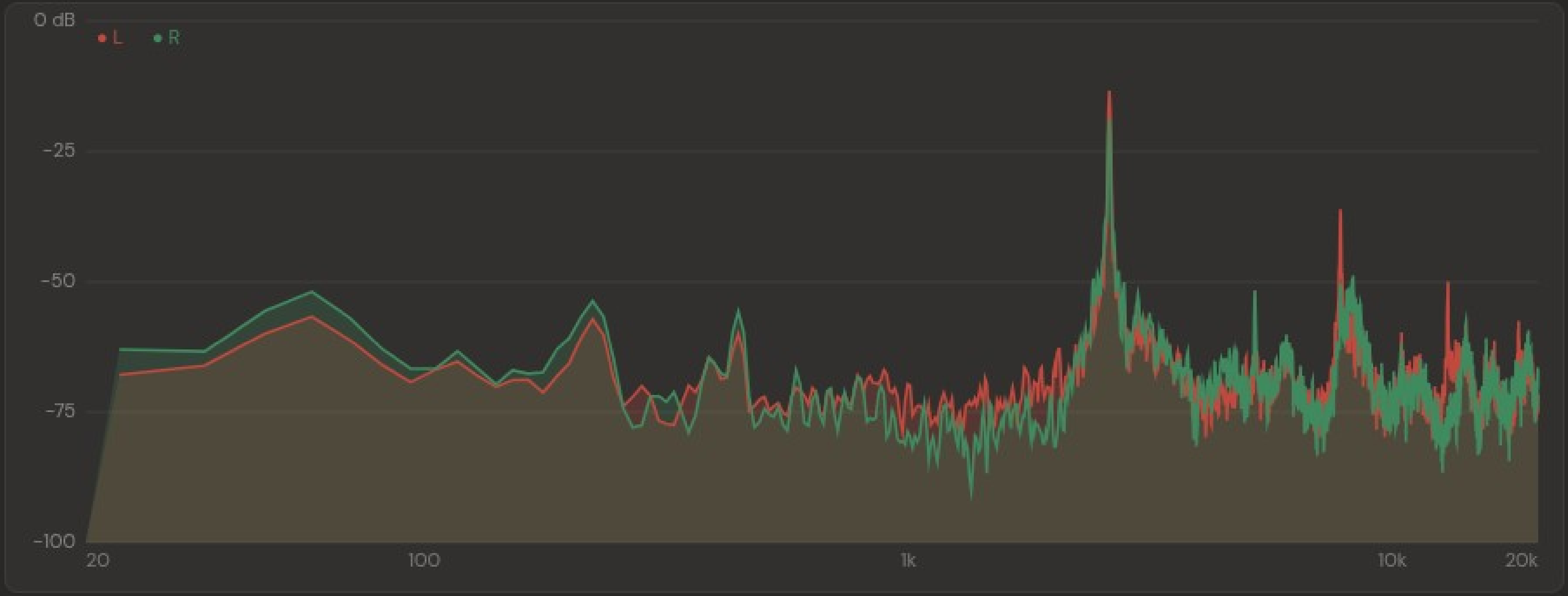
$$P[k] = \frac{|X[k]|^2}{N^2}$$

$$P_{dB}[k] = 10 \cdot \log_{10}\left(\frac{P[k]}{P_{ref}}\right)$$

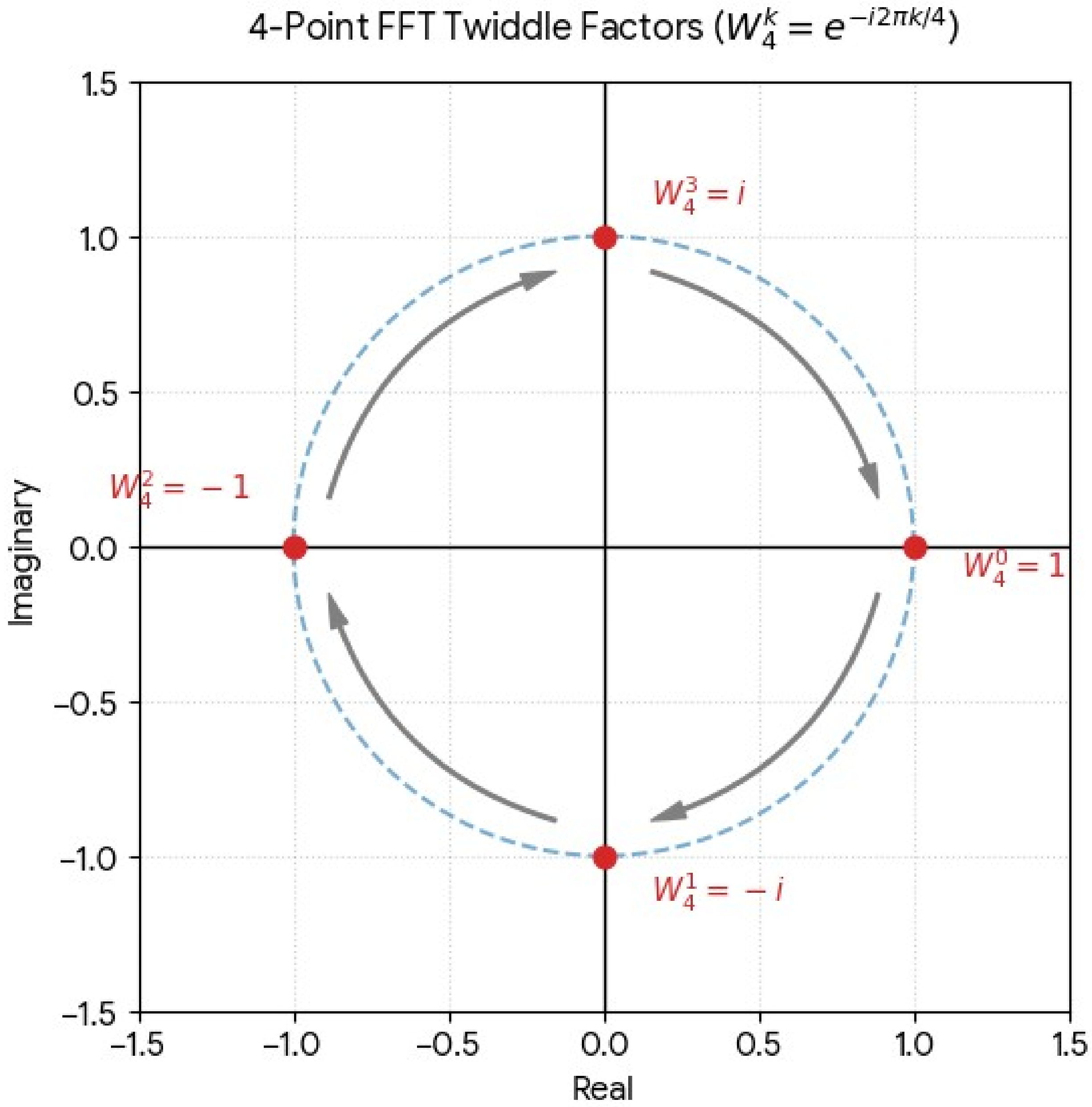
DOMINANT FREQUENCY	NOTE	CORRELATION
2.60 kHz	E7 -25 cents	0.92

Spectrum | Spectrogram

Freeze | Screenshot



Analyzing microphone input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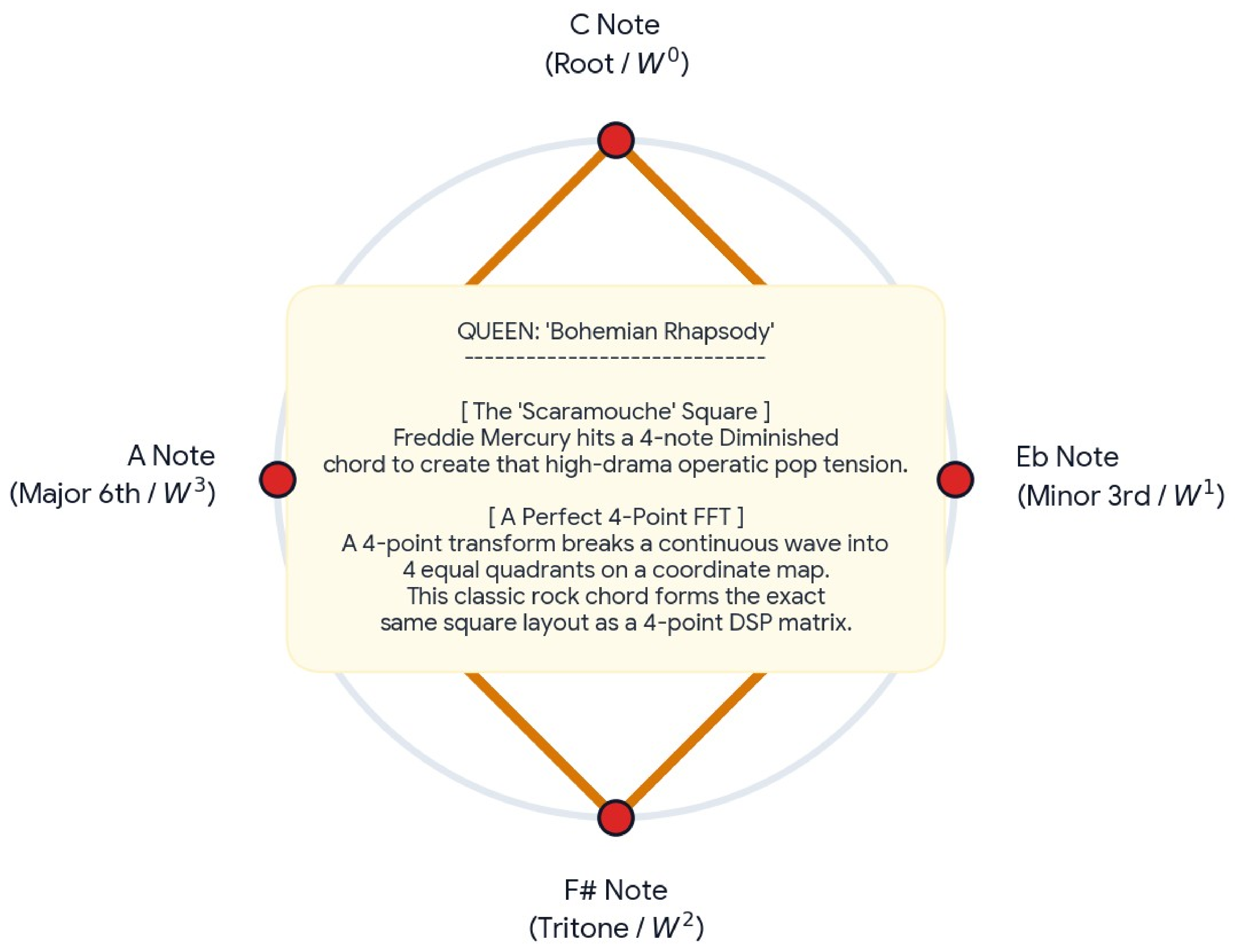
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5


C Note
(Root / W^0)
A Note
(Major 6th / W^3)
Eb Note
(Minor 3rd / W^1)
F# Note
(Tritone / W^2)
QUEEN: 'Bohemian Rhapsody'
[The 'Scaramouche' Square]
Freddie Mercury hits a 4-note Diminished chord to create that high-drama operatic pop tension.
[A Perfect 4-Point FFT]
A 4-point transform breaks a continuous wave into 4 equal quadrants on a coordinate map. This classic rock chord forms the exact same square layout as a 4-point DSP matrix.

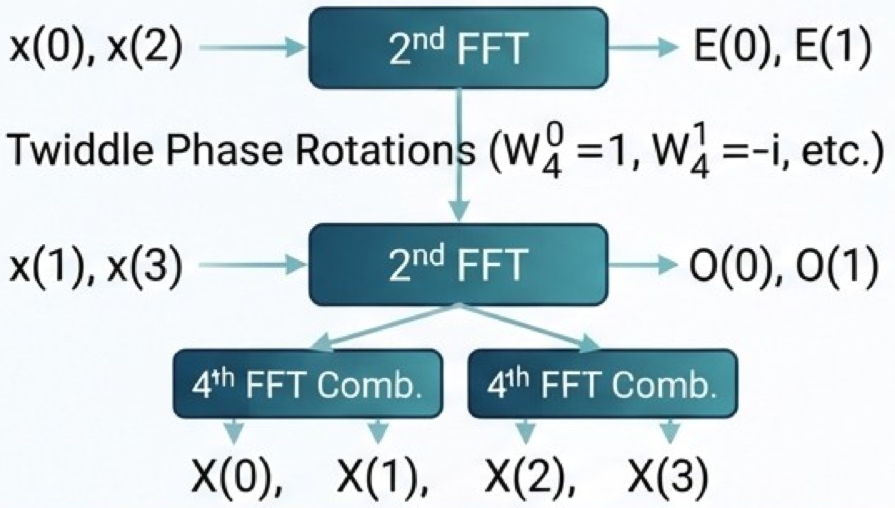
The operatic section kicks off right at the **3:03 mark** of the track.

The entire rock ballad completely cuts away, and the layered vocals transition the listener into the opera. The exact lyric where the 4-point symmetric square layout snaps into full effect is at **3:09**, right when they belt out **"Scaramouche, Scaramouche, will you do the Fandango?"**

Looking Ahead: The Quantum Fourier Transform (QFT)

Classical 4-Point FFT (Butterfly Network)

(Butterfly Network)



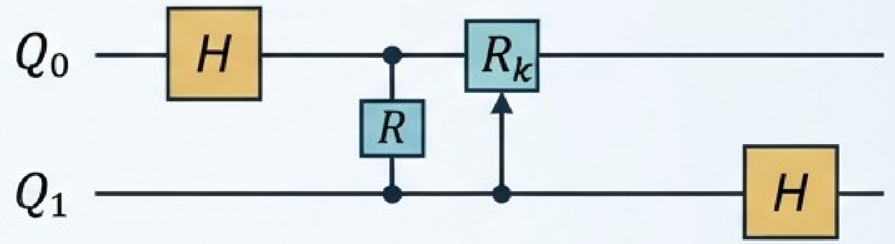
Input Array Size: 4 elements = 4 Dimensions.
Data: Complex Time-Domain Samples.

4 Dimensions = 2 Qubits: Native Exponential Scaling

2-Qubit Quantum State Space (Entangled)

(Entangled)

$$|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$$



$$|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$$

Qubit Space Size: 2^2 = 4 Dimensions.
Data: Complex Probability Amplitudes.

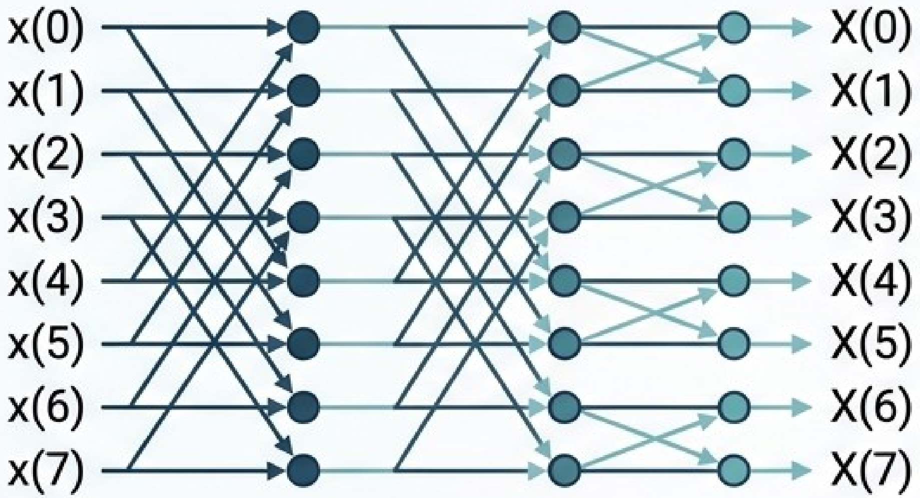
Preview for Week 15: The Exponential Advantage

1. **Hardware Parallels:** Classical butterfly network operations (Sum/Difference) and twiddle factor rotations map directly to fundamental Quantum Gates (Hadamard, Controlled-Phase).
2. **Computational Speedup:** Running a native butterfly network inside exponentially scalable entangled states allows the QFT (Quantum Fourier Transform) to achieve an exponential speedup over classical FFT algorithms (e.g., Shor's Algorithm (e.g., Shor's Algorithm).

🚀 Looking Ahead: The Quantum Fourier Transform (QFT)

Classical 8-Point FFT (Butterfly Network)

(Butterfly Network)



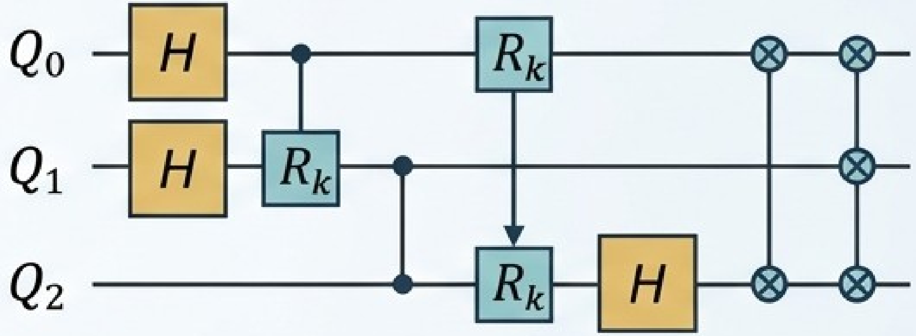
Input Array Size: 8 elements = 8 Dimensions.
Data: Complex Time-Domain Samples.

8 Dimensions = 3 Qubits: Native Exponential Scaling

3-Qubit Quantum State Space (Entangled)

(Entangled)

$$|\psi\rangle = c_0|000\rangle + c_1|001\rangle + ... + c_7|111\rangle$$



$$|\psi\rangle = c_0|000\rangle + c_1|001\rangle + \cdots + c_7|111\rangle$$

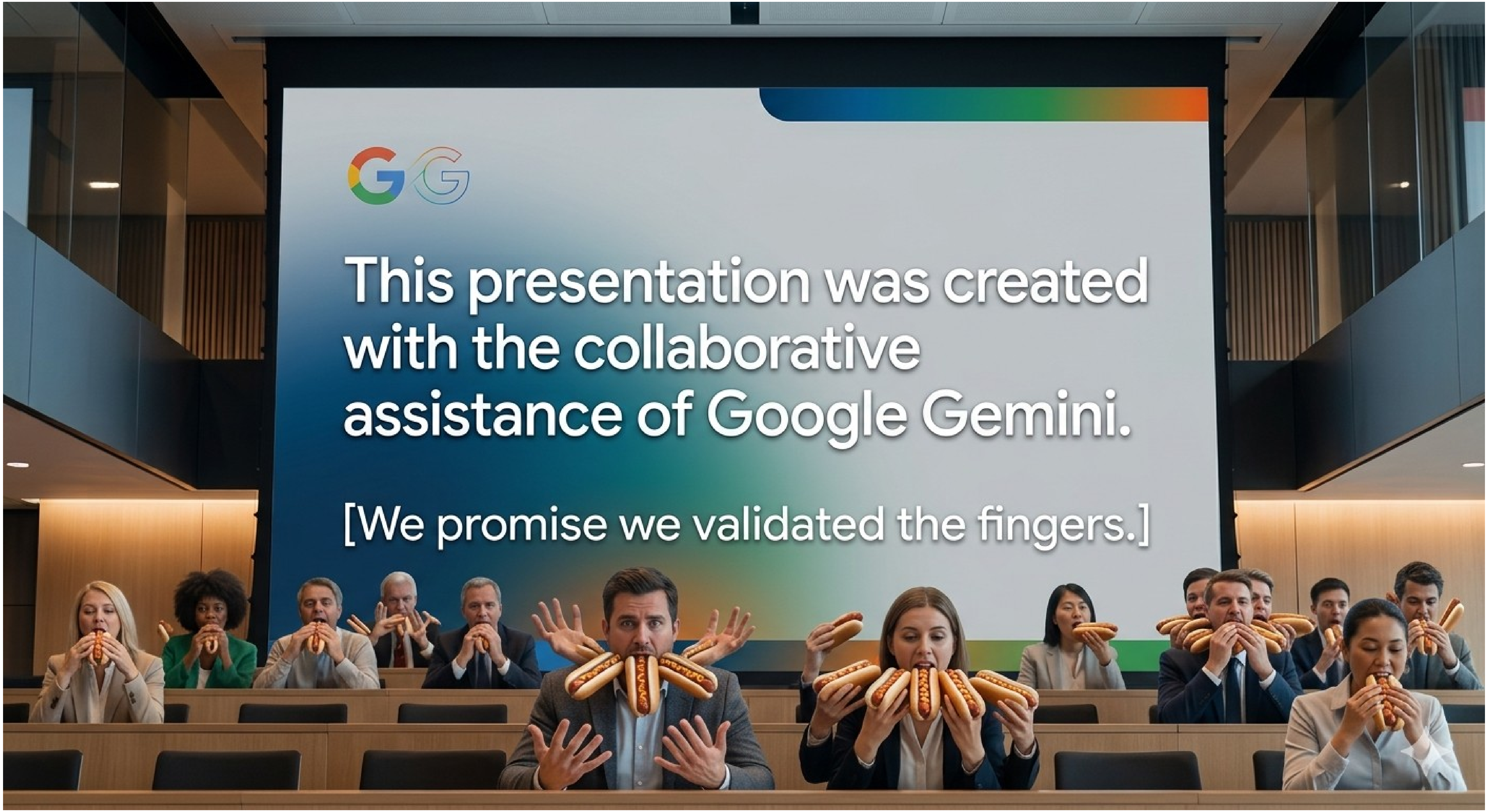
Qubit Space Size: 2^3 = 8 Dimensions.
Data: Complex Probability Amplitudes.

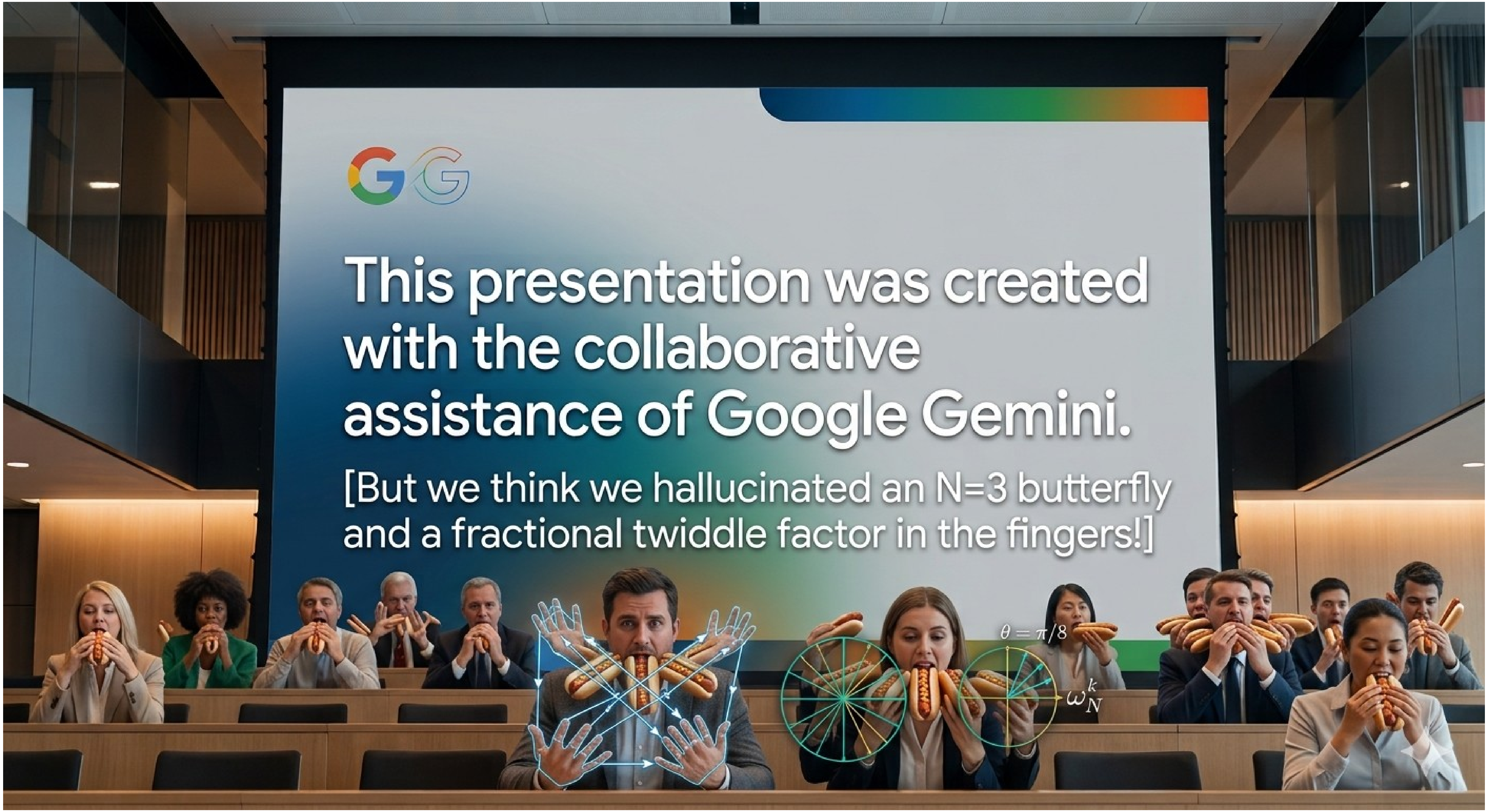
🎓 Preview for Week 15: The Exponential Advantage

1. **Hardware Parallels:** Classical butterfly network operations (Sum/Difference) and twiddle factor rotations map directly to fundamental Quantum Gates (Hadamard, Controlled-Phase).
2. **Computational Speedup:** Running a native butterfly network inside exponentially scalable entangled states allows the QFT (Quantum Fourier Transform) to achieve an exponential speedup over classical FFT algorithms (e.g., Shor's Algorithm (e.g., Shor's Algorithm).

How to prepare for the FFT on the exam and pop quiz

- Check Canvas, I’ll post example exams and solutions
- *The Nature of Computation*, by Cris Moore and Stephan Mertens, pages 49–53
- Tons of YouTube videos


This presentation was created with the collaborative assistance of Google Gemini.
[We promise we validated the fingers.]


This presentation was created with the collaborative assistance of Google Gemini.
[But we think we hallucinated an N=3 butterfly and a fractional twiddle factor in the fingers!]
$\theta = \pi/8$
ω_N^k