Information theory and PRNG

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What do we need randomness for?

- Private keys
- Session keys
- Initialization Vectors
- Random padding
- ...

Don’t do this...

```
int i = 10000000 + new Random().nextInt(89999999);
int j = 10000000 + new Random().nextInt(89999999);
return (String.valueOf(i) + String.valueOf(j)).getBytes();
```

Figure 1: Decompiled Java method generating an AES session key in version 6.3.0.1920.

```
Random random = new Random(System.currentTimeMillis());
byte[] bArr = new byte[8];
byte[] bArr2 = new byte[8];
random.nextBytes(bArr);
random.nextBytes(bArr2);
return new SecretKeySpec(ByteUtils.mergeByteData(bArr, bArr2), "AES");
```

Figure 2: Decompiled Java method generating an AES session key in version 6.5.0.2170.

https://arxiv.org/pdf/1802.03367.pdf

Requirements (Shannon, 1948)

1) $I(p) \geq 0$ (information is non-negative, $p \geq 1$)

2) $I(1) = 0$ (events that always occur carry no information)

3) $I(p1\ p2) = I(p1) + I(p2)$ (information due to independent events is additive)

Also, continuity, symmetry, and maximum when all possible events are equiprobable.

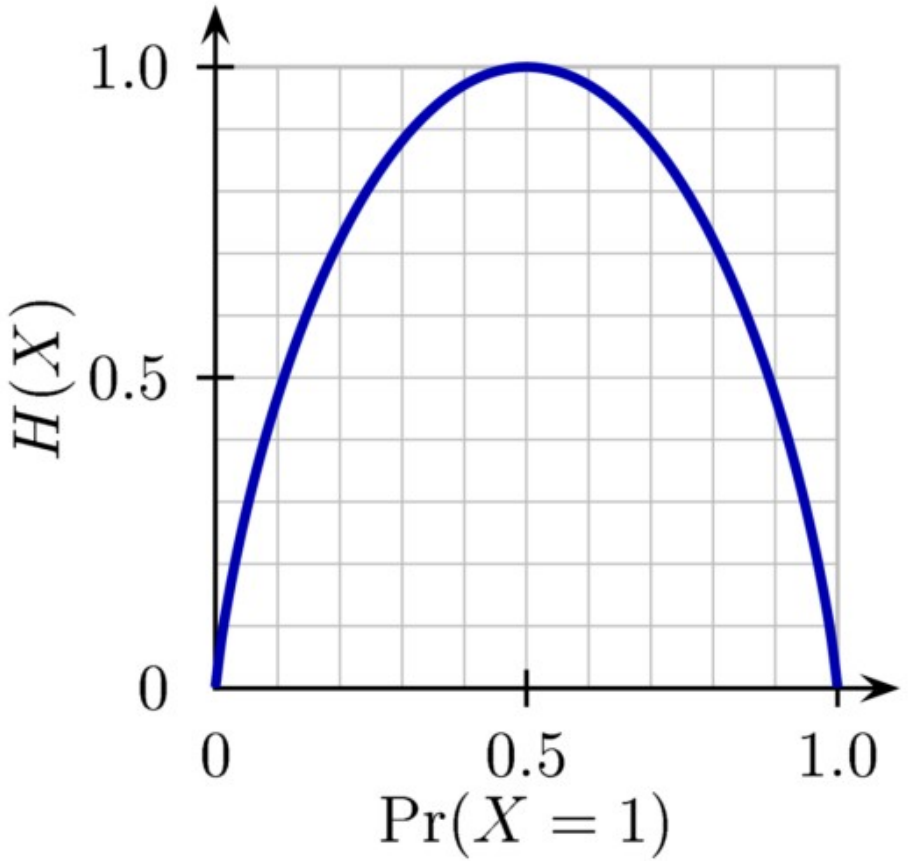
$$I(p) = \log(1/p)$$

1) $I(1/2) = 0.30102999566...$
2) $I(1) = 0$
3) $I(1/2) + I(1/3) = \log(2) + \log(3) = 0.77815125038...$
Joint probability: $I(1/6) = 0.77815125038...$

Continuity: $I(1/2.01) = 0.30319605742...$
Symmetry: $\log(3) + \log(2) = 0.77815125038...$
Maximum: $\log 3 + \log 3 + \log 3 = 1.43136376416...$
$\log 2 + \log 4 + \log 4 = 1.20411998266...$

Information = Entropy = Surprise

$$H[p] = -\sum_{i=1}^{k} p_i \log p_i$$



Pop quiz #1

- When a 3yo walks by with a stepstool…
  - 4 times out of 10 it's to get something they're not supposed to have
  - 2 times out of 10 it's to climb up to somewhere they're not supposed to be
  - 1 time out of 10 it's to wash their hands
  - 1 time out of 10 it's to get something they're allowed to have
  - 1 time out of 10 it's to use as a dollhouse
  - 1 time out of 10 it's to turn over and use as a storage bin
- What is the entropy of each instance of 3yo stepstool habits?

Answer

Input:

$$-0.4 \log_2(0.4) - 0.2 \log_2(0.2) - 4\,(0.1 \log_2(0.1))$$

Result:

$$2.32193\ldots$$

Pop quiz #2

- There are three possible states the Tempe weather could be in during any given hour on a summer day (very hot and bright out, very hot and it’s nighttime, monsoonal rains). What probability distribution over these events would give the maximum entropy in terms of what you might observe in a randomly chosen hour from the summer?

Answer

Input:

$$-\frac{1}{3}\log_2\left(\frac{1}{3}\right)-\frac{1}{3}\log_2\left(\frac{1}{3}\right)-\frac{1}{3}\log_2\left(\frac{1}{3}\right)$$

Exact result:

$$\frac{\log(3)}{\log(2)}$$

$\log(x)$ is the natural logarithm

Decimal approximation:

More digits

1.584962500721156181453738943947816508759814407692481060455...

Entropy

- A measure of the information of a **random process**
- Pop quiz #3: Based on the above definition, order the following binary sequences from most entropy to least entropy?:

  A) 11111111000000000000000011111111

  B) 10111001110011001100010000110100

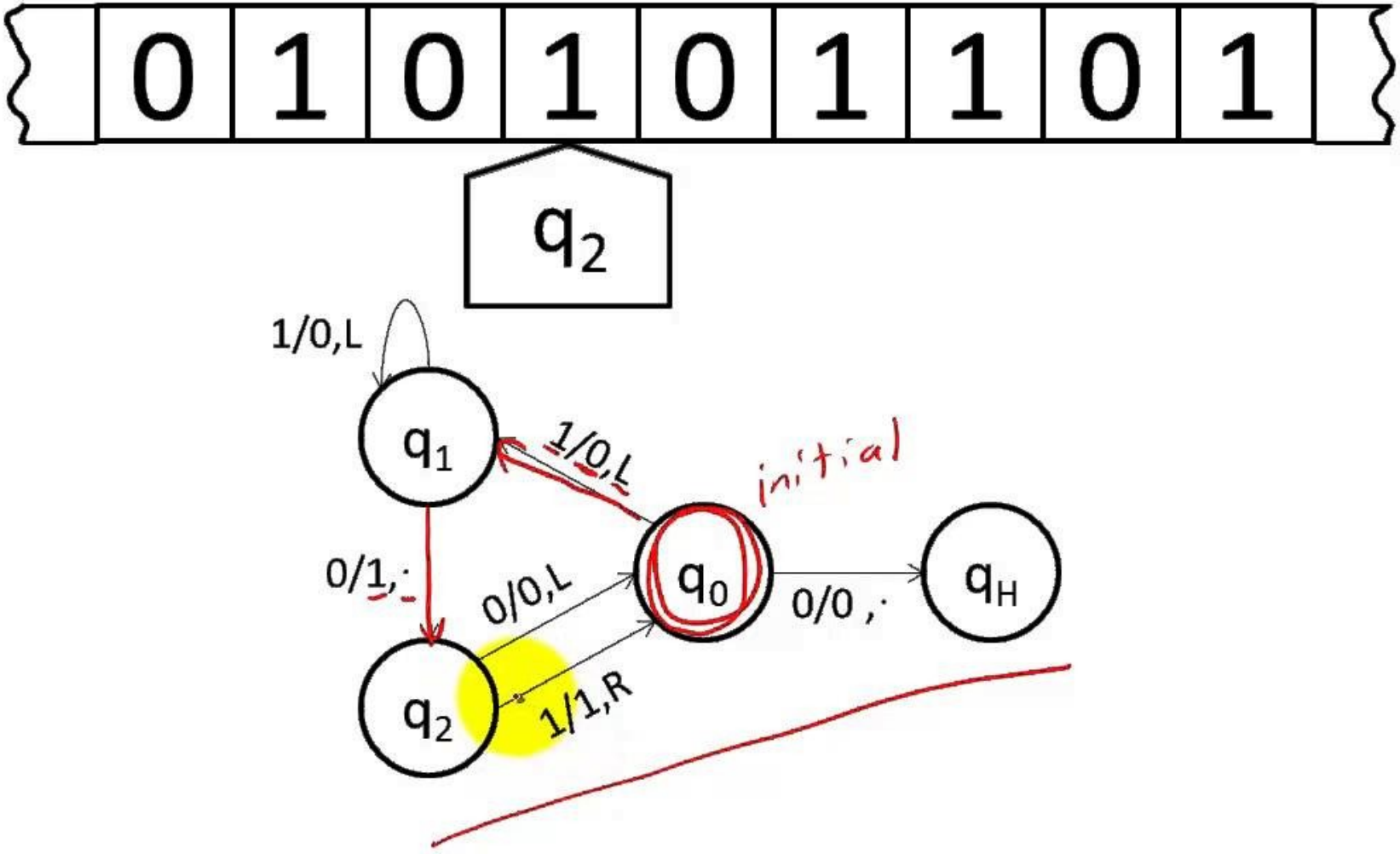
  C) 00000000000000000000000000000000

  D) 00010010000000100000000100000001

I pity the fool who uses the word “entropy” to describe a bit sequence or string without realizing that they are implicitly talking about algorithmic entropy (*a.k.a.*, Kolmogorov complexity) rather than the standard definition of entropy that Claude Shannon used to describe random processes!

Turing machines

Source: https://www.youtube.com/watch?v=gJQTFhkhwPA



Halting problem

- From a description of an arbitrary computer program and an input, determine whether the program will finish running or continue to run forever.

  P: if (halts(P)) then: while true {}; else halt;

Gödel's Incompleteness Theorem

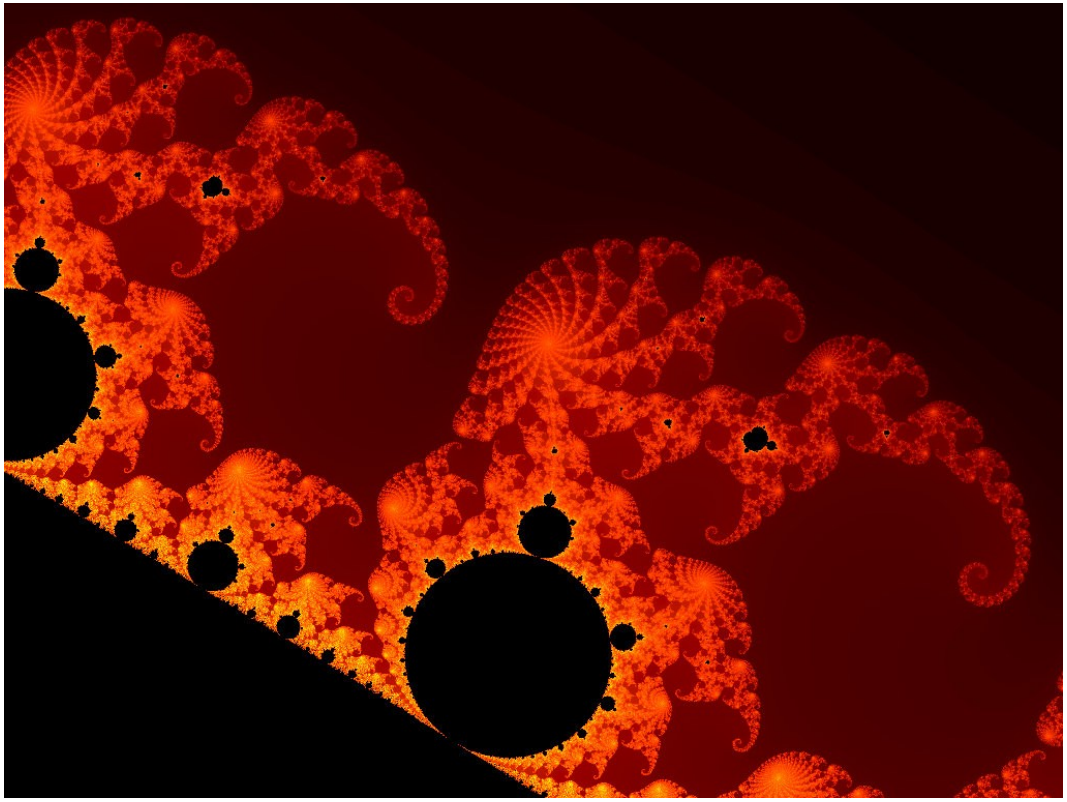
- In axiomatic systems capable of arithmetic, can't separate mathematics and meta-mathematics, can always form sentences of the form:

  This sentence is false.

- As Einstein put it:
  - “As far as the laws of mathematics refer to reality, they are not certain, as far as they are certain, they do not refer to reality.”
  - “We can't solve problems by using the same kind of thinking we used when we created them.”

Kolmogorov complexity

- Defined as the length of the shortest computer program that produces the object as output.



Chaitin's incompleteness theorem

- “The fact that a specific string is complex cannot be formally proven, if the complexity of the string is above a certain threshold.” [Wikipedia]
- Berry's paradox:
  - “the smallest positive integer not definable in fewer than twelve words”

PRNG

- Example:

  $X_{i+1} = X_i * a + b \bmod m$

- Seed
- Period
- Cryptographically strong
  - Attacker knows the algorithm and lots of past bits
  - *E.g.*, take the above and encrypt it with AES and a well chosen key.
  - For more detailed examples, look up Yarrow or Fortuna.

PRNG problems

- Seeding problems
  - E.g., when you first boot an embedded device
- Periodicity problems
  - Never use srand(time) and rand() for crypto
- Backdoors
  - Dual_EC_DRBG
- Predictability
  - Witty worm

Linux kernel's entropy pool

Secure hash function (e.g., SHA1 or ChaCha20-based)

`/dev/urandom`
`/dev/random`

You have 12 coins, one is counterfeit. The counterfeit is either slightly heavier or slightly lighter, otherwise it's impossible to tell. You have a balance. Using the balance the fewest number of times, find the counterfeit coin.

Sources

- https://en.wikipedia.org/wiki/Entropy_%28information_theory%29
- http://www.amazon.com/Elements-Information-Theory-Telecommunications-Processing/dp/0471241954/ref=sr_1_8?ie=UTF8&qid=1457368787&sr=8-8&keywords=information+theory
- http://www.amazon.com/Mathematical-Theory-Communication-Claude-Shannon/dp/0252725484/ref=sr_1_1?s=books&ie=UTF8&qid=1457368808&sr=1-1&keywords=information+theory
- https://en.wikipedia.org/wiki/Turing_machine
- https://www.youtube.com/watch?v=gJQTFhkhwPA
- https://en.wikipedia.org/wiki/Halting_problem
- http://www.icir.org/vern/papers/witty-imc05.pdf