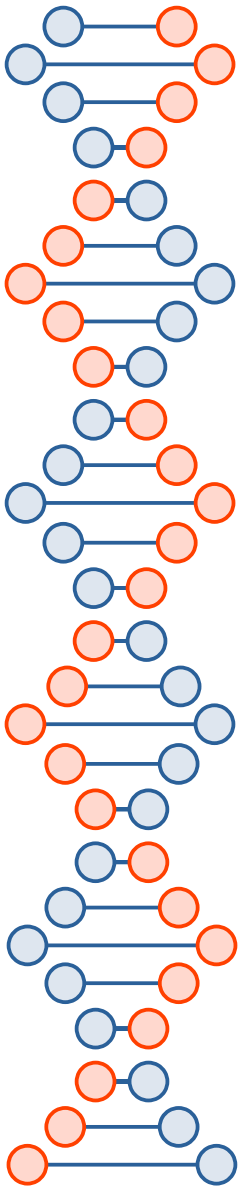
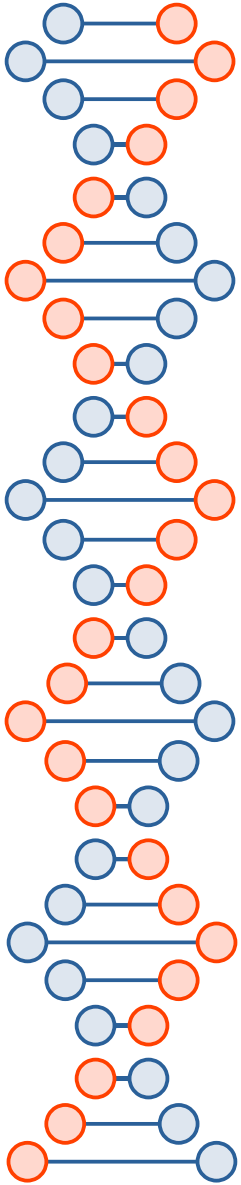


RSA

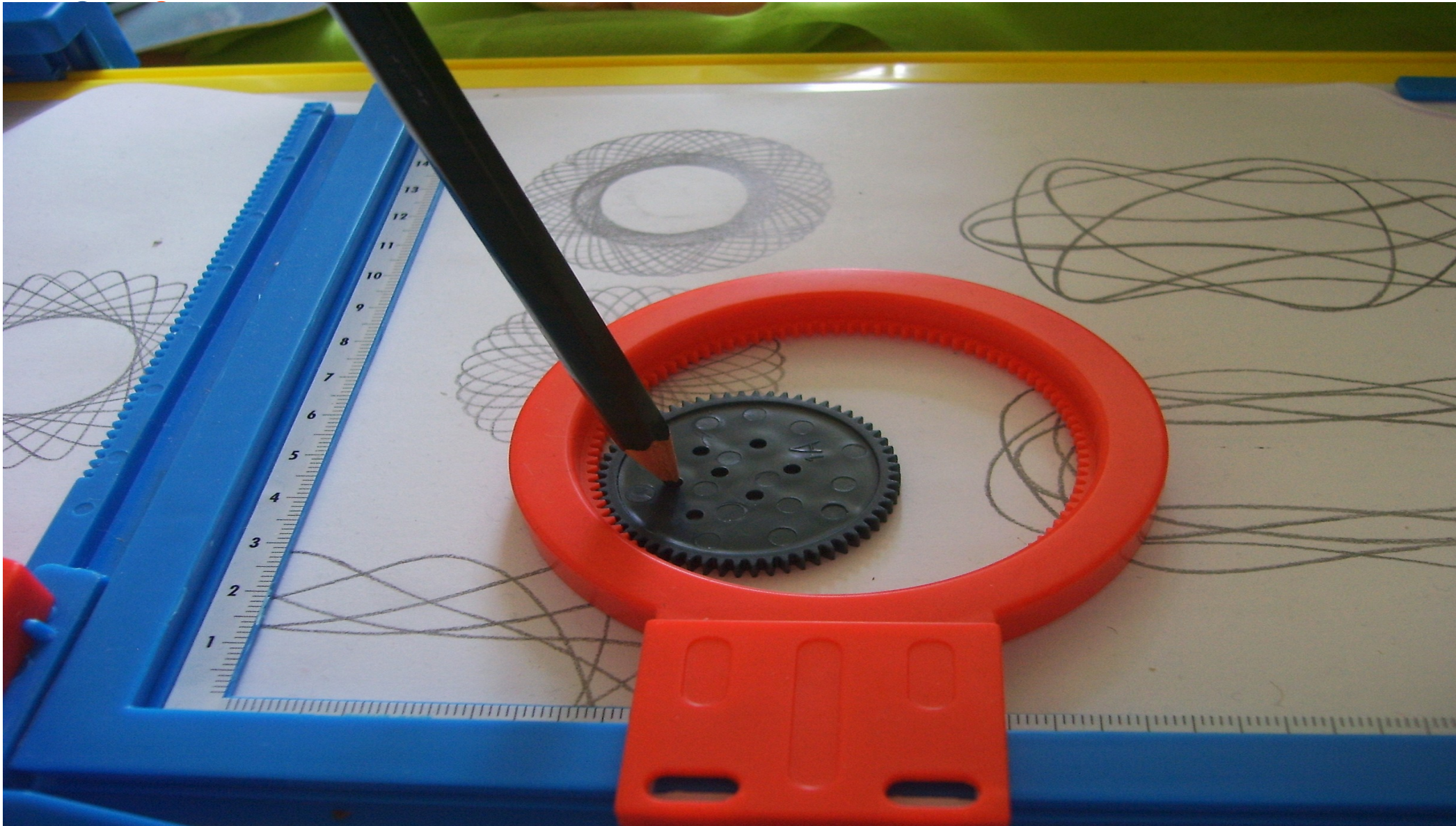
CSE 539

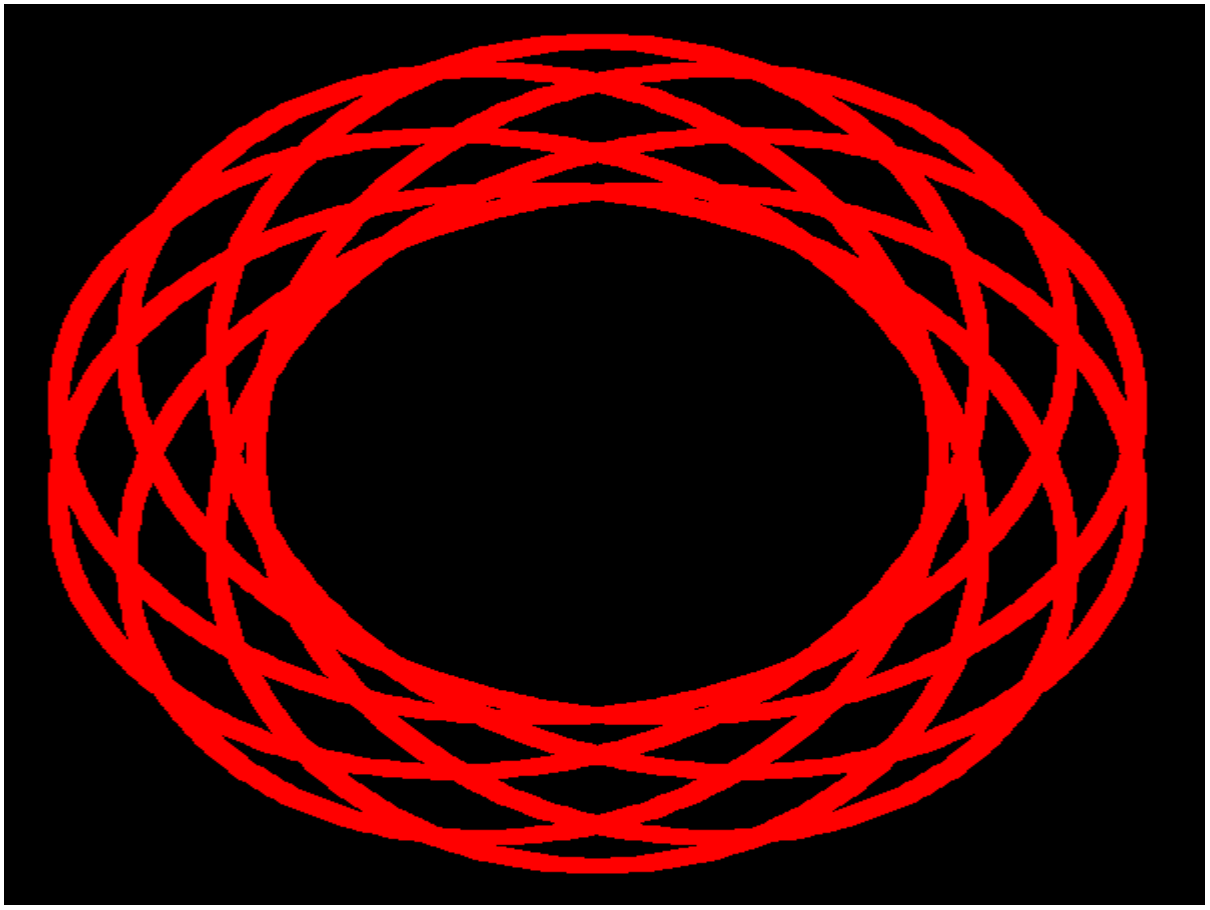
jedimaestro@asu.edu

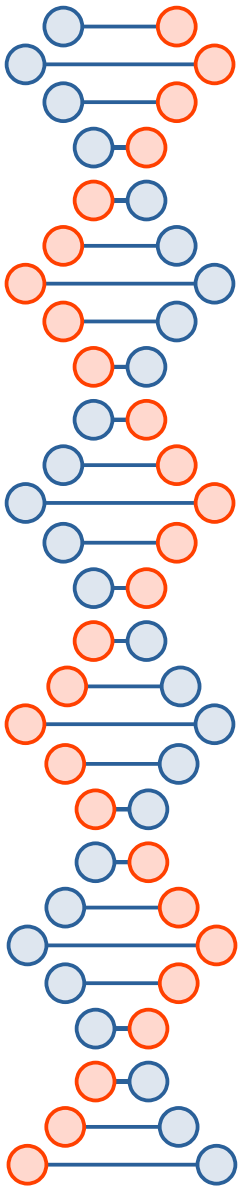


RSA *vs.* DH

- Diffie-Hellman (1976)
  - Key exchange
  - *Both sides get to choose something random*
- RSA (1977)
  - Encryption
  - Signatures

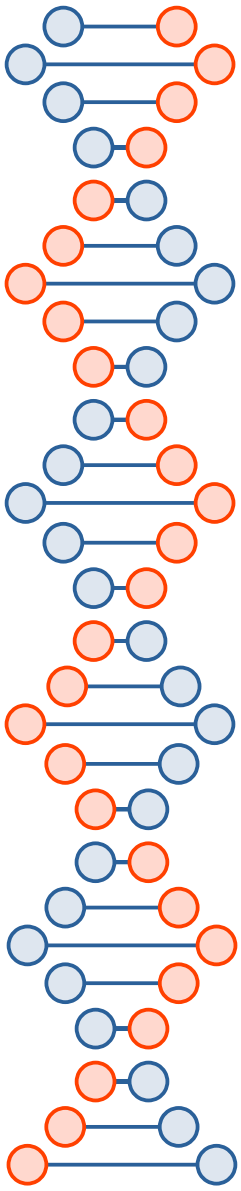






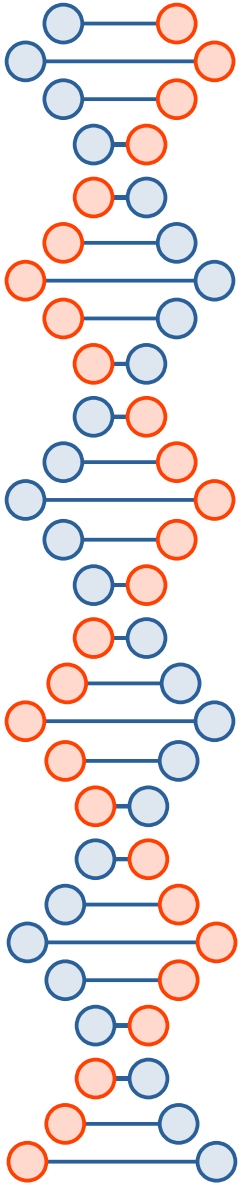
RSA

- Security is based on the hardness of integer factorization

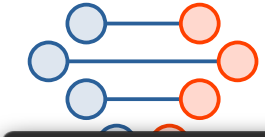


n = pq

- p and q are primes, suppose p = 61, q = 53
- n = 3233
- Euler's totient counts the positive integers up to n that are relatively prime to n
- totient(n) = lcm(p – 1, q – 1) = 780
  - 52,104,156,208,260,312,364,416,468,520,572,624,676,728,780
  - 60,120,180,240,300,360,420,480,540,600,660,720,780
- Choose 1 < e < 780 coprime to 780, e.g., e = 17
- d is the modular multiplicative inverse of e, d = 413
- 413 * 17 mod 780 = 1

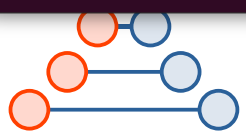


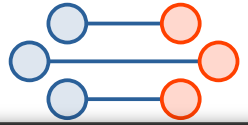
- Public key is (n = 3233, e = 17)
- Private key is (n = 3233, d = 413)
- Encryption: $c(m = 65) = 65^{17} \bmod 3233 = 2790$
- Decryption: $m = 2790^{413} \bmod 3233 = 65$
- Could also do...
  - Signature: $s = 100^{413} \bmod 3233 = 1391$
  - Verification: $100 = 1391^{17} \bmod 3233$
- Fast modular exponentiation is the trick
- Using RSA for key exchange or encryption is often a red flag, more commonly used for signatures



jedi@route66: ~

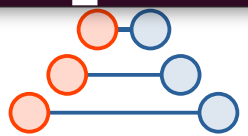
```
jedi@route66:~$ python3
Python 3.8.2 (default, Jul 16 2020, 14:00:26)
[GCC 9.3.0] on linux
Type "help", "copyright", "credits" or "license" for more information.
>>> for i in range (52, 781, 52):
...     for j in range (60, 781, 60):
...             if (i == j):
...                     print(i)
...
780
>>> print((413 * 17) % 780)
1
>>> print(pow(2790, 413, 3233))
65
>>> print(pow(65, 17, 3233))
2790
>>> print(pow(100, 413, 3233))
1391
>>> print(pow(1391, 17, 3233))
100
>>>
```

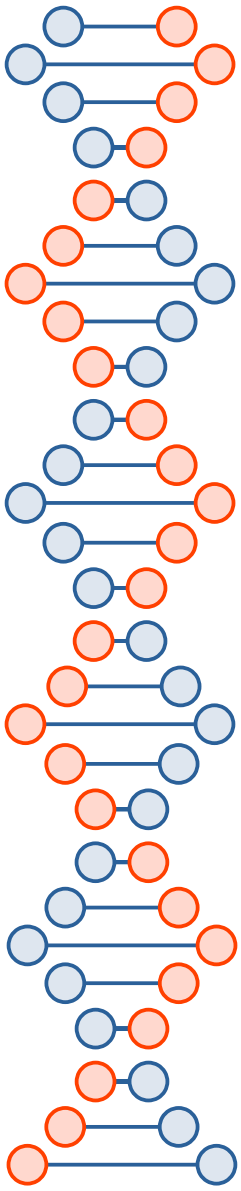




jedi@route66: ~

```
1
>>> print(pow(2790, 413, 3233))
65
>>> print(pow(65, 17, 3233))
2790
>>> print(pow(100, 413, 3233))
1391
>>> print(pow(1391, 17, 3233))
100
>>> print(pow(7, 17, 3233))
2369
>>> print((2369*2790) % 3233)
1258
>>> print(pow(1258, 413, 3233))
455
>>> print(7*65)
455
>>> print("{0:b}".format(78913))
1001101000100001
>>> print("{0:b}".format(78913*32))
10011010001000001000000
>>> print("{0:b}".format(78913<<5))
10011010001000001000000
>>>
```





“Relatively prime”

- 9 is not prime, $9 = 3^2$
- 13 is prime
- 10 is not prime, 10 = 5*2
- 9 and 10 are relatively prime, gcd(9,10) = 1
- 5 and 10 are not relatively prime, gcd(5,10) = 5
- Also called “coprime”

$$M^{\phi(n)} \equiv 1 \pmod{n} \ . \tag{3}$$

Here $\phi(n)$ is the Euler totient function giving number of positive integers less than n which are relatively prime to n. For prime numbers p,

$$\phi(p) = p - 1 \ .$$

In our case, we have by elementary properties of the totient function [7]:

$$\begin{aligned} \phi(n) &= \phi(p) \cdot \phi(q) \\ &= (p-1) \cdot (q-1) \\ &= n - (p+q) + 1 \ . \end{aligned} \tag{4}$$

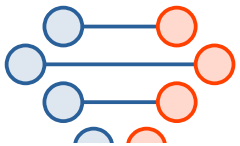
Since d is relatively prime to $\phi(n)$, it has a multiplicative inverse e in the ring of integers modulo $\phi(n)$:

Euler's totient function

- https://en.wikipedia.org/wiki/Euler%27s_totient_function

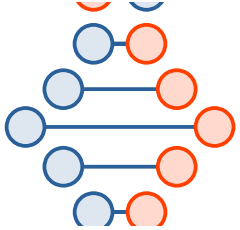
In number theory, **Euler's totient function** counts the positive integers up to a given integer n that are relatively prime to n. It is written using the Greek letter phi as $\varphi(n)$ or $\phi(n)$, and may also be called **Euler's phi function**.

Euler's totient function is a multiplicative function, meaning that if two numbers m and n are relatively prime, then $\varphi(mn) = \varphi(m)\varphi(n)$.[4][5] This function gives the order of the multiplicative group of integers modulo n (the group of units of the ring $\mathbb{Z}/n\mathbb{Z}$).[6] It is also used for defining the RSA encryption system.



Since d is relatively prime to $\phi(n)$, it has a multiplicative inverse e in the ring of integers modulo $\phi(n)$:

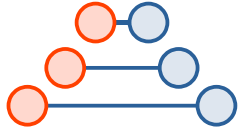
$$e \cdot d \equiv 1 \pmod{\phi(n)}. \tag{5}$$



$$\begin{aligned} D(E(M)) &\equiv (E(M))^d \equiv (M^e)^d \pmod{n} = M^{e \cdot d} \pmod{n} \\ E(D(M)) &\equiv (D(M))^e \equiv (M^d)^e \pmod{n} = M^{e \cdot d} \pmod{n} \end{aligned}$$

and

$$M^{e \cdot d} \equiv M^{k \cdot \phi(n)+1} \pmod{n} \text{ (for some integer } k).$$



From (3) we see that for all M such that p does not divide M

$$M^{p-1} \equiv 1 \pmod{p}$$

and since $(p-1)$ divides $\phi(n)$

$$M^{k\cdot\phi(n)+1} \equiv M \pmod{p}.$$

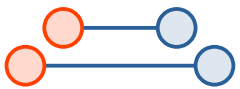
This is trivially true when $M \equiv 0 \pmod{p}$, so that this equality actually holds for *all* M. Arguing similarly for q yields

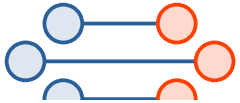
$$M^{k\cdot\phi(n)+1} \equiv M \pmod{q}\ .$$

Together these last two equations imply that for all M,

$$M^{e\cdot d} \equiv M^{k\cdot\phi(n)+1} \equiv M \pmod{n}.$$

This implies (1) and (2) for all $M, 0 \leq M < n$. Therefore E and D are inverse permutations. (We thank Rich Schroeppel for suggesting the above improved version of the authors' previous proof.)

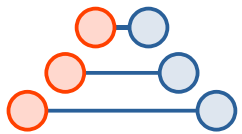


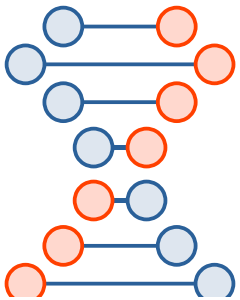


Computing $M^e \pmod{n}$ requires at most $2 \cdot \log_2(e)$ multiplications and $2 \cdot \log_2(e)$ divisions using the following procedure (decryption can be performed similarly using d instead of e):

Step 1. Let $e_k e_{k-1} \ldots e_1 e_0$ be the binary representation of e.
Step 2. Set the variable C to 1.
Step 3. Repeat steps 3a and 3b for $i = k, k-1, \ldots, 0$:
  Step 3a. Set C to the remainder of C^2 when divided by n.
  Step 3b. If $e_i = 1$, then set C to the remainder of $C \cdot M$ when divided by n.
Step 4. Halt. Now C is the encrypted form of M.

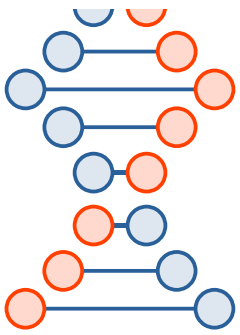
This procedure is called "exponentiation by repeated squaring and multiplication." This procedure is half as good as the best; more efficient procedures are known. Knuth [3] studies this problem in detail.

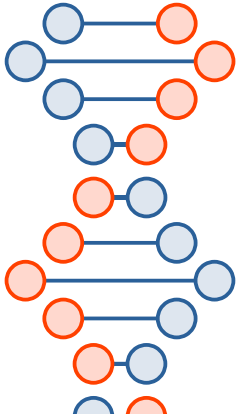




Each user must (privately) choose two large random numbers p and q to create his own encryption and decryption keys. These numbers must be large so that it is not computationally feasible for anyone to factor $n = p \cdot q$. (Remember that n, but not p or q, will be in the public file.) We recommend using 100-digit (decimal) prime numbers p and q, so that n has 200 digits.

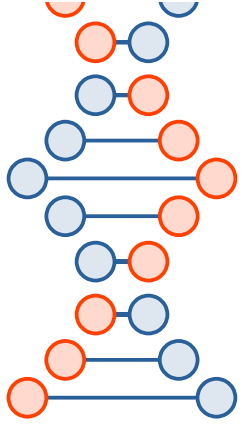
To find a 100-digit "random" prime number, generate (odd) 100-digit random numbers until a prime number is found. By the prime number theorem [7], about $(\ln 10^{100})/2 = 115$ numbers will be tested before a prime is found.

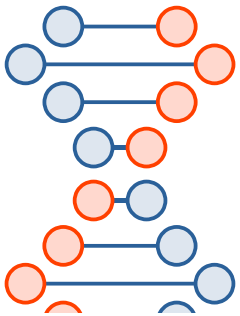




To test a large number b for primality we recommend the elegant "probabilistic" algorithm due to Solovay and Strassen [12]. It picks a random number a from a uniform distribution on $\{1, \ldots, b-1\}$, and tests whether

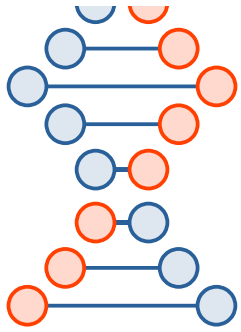
$$\gcd(a, b) = 1 \text{ and } J(a, b) = a^{(b-1)/2} \pmod{b}, \tag{6}$$

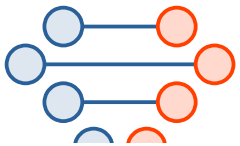




To gain additional protection against sophisticated factoring algorithms, p and q should differ in length by a few digits, both $(p-1)$ and $(q-1)$ should contain large prime factors, and $\gcd(p-1, q-1)$ should be small. The latter condition is easily checked.

To find a prime number p such that $(p-1)$ has a large prime factor, generate a large random prime number u, then let p be the first prime in the sequence $i \cdot u + 1$, for $i = 2, 4, 6, \ldots$. (This shouldn't take too long.) Additional security is provided by ensuring that $(u-1)$ also has a large prime factor.

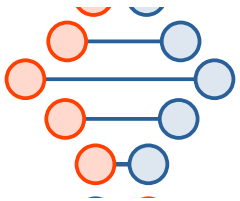




C How to Choose d

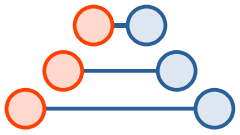
It is very easy to choose a number d which is relatively prime to $\phi(n)$. For example, any prime number greater than $\max(p, q)$ will do. It is important that d should be chosen from a large enough set so that a cryptanalyst cannot find it by direct search.

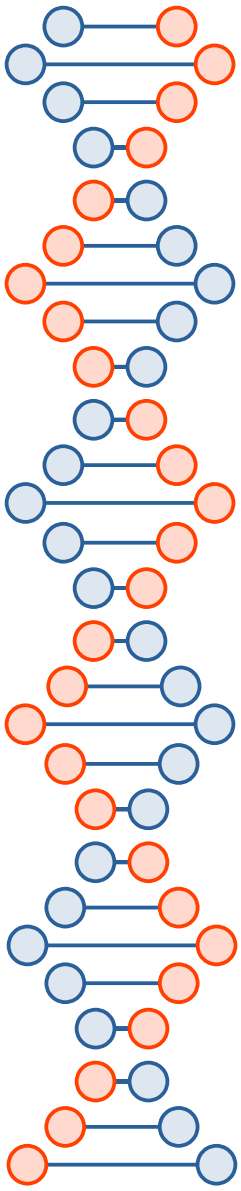
D How to Compute e from d and $\phi(n)$



Euclid's algorithm

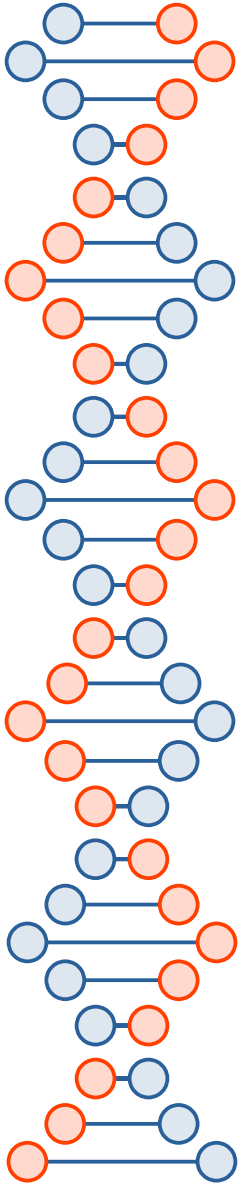
If e turns out to be less than $\log_2(n)$, start over by choosing another value of d. This guarantees that every encrypted message (except $M = 0$ or $M = 1$) undergoes some "wrap-around" (reduction modulo n) .





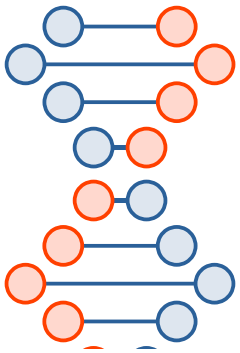
An “exercise”

- You don’t have to turn this in
- Go to a few of your favorite websites
- If they don’t support HTTPS, say so in Canvas
  - They really should support HTTPS, I’d be interested to know major websites that still don’t
- If they do support HTTPS, check the public key’s exponent using your browser’s ability to inspect a TLS cert
  - Is it 0x10001? (65537 in decimal)



Takeaways so far

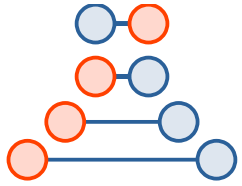
- RSA let’s you do encryption, signatures
- Even textbook RSA is not trivial to implement

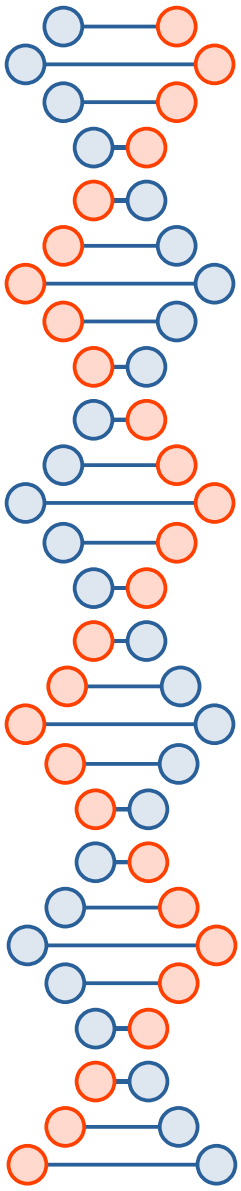


The era of "electronic mail" [10] may soon be upon us;

encryption keys. (We assume that the intruder cannot modify or insert messages into the channel.) Ralph Merkle has developed another solution [5] to this problem.

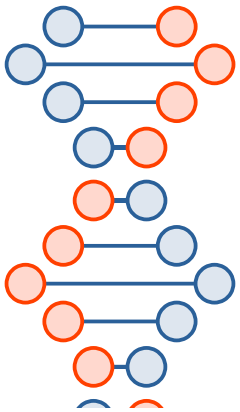
A public-key cryptosystem can be used to "bootstrap" into a standard encryption scheme such as the NBS method. Once secure communications have been established, the first message transmitted can be a key to use in the NBS scheme to encode all following messages. This may be desirable if encryption with our method is slower than with the standard scheme. (The NBS scheme is probably somewhat faster if special-purpose hardware encryption devices are used; our scheme may be faster on a general-purpose computer since multiprecision arithmetic operations are simpler to implement than complicated bit manipulations.)



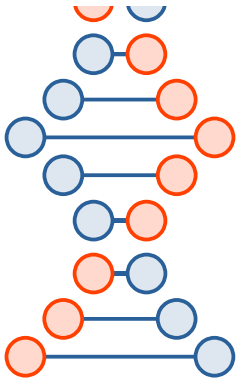


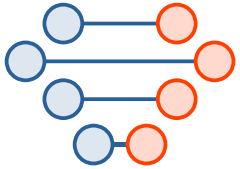
200-digit message M

$\log_2(10^{200})$ = about 665 bits



Since no techniques exist to *prove* that an encryption scheme is secure, the only test available is to see whether anyone can think of a way to break it. The NBS standard was "certified" this way; seventeen man-years at IBM were spent fruitlessly trying to break that scheme. Once a method has successfully resisted such a concerted attack it may for practical purposes be considered secure. (Actually there is some controversy concerning the security of the NBS method [2].)

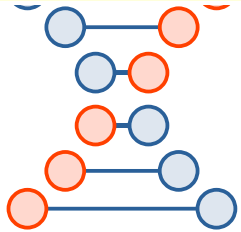


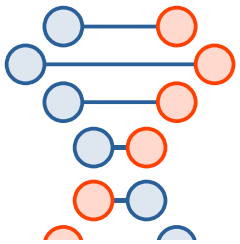


How can n be factored using $\phi(n)$? First, $(p+q)$ is obtained from n and $\phi(n) = n-(p+q)+1$. Then $(p-q)$ is the square root of $(p+q)^2-4n$. Finally, q is half the difference of $(p+q)$ and $(p-q)$.

Therefore breaking our system by computing $\phi(n)$ is no easier than breaking our system by factoring n. (This is why n must be composite; $\phi(n)$ is trivial to compute if n is prime.)

A knowledge of d enables n to be factored as follows. Once a cryptanalyst knows d he can calculate $e \cdot d - 1$, which is a multiple of $\phi(n)$. Miller [6] has shown that n can be factored using any multiple of $\phi(n)$. Therefore if n is large a cryptanalyst should not be able to determine d any easier than he can factor n.

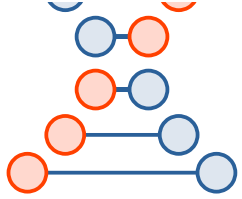


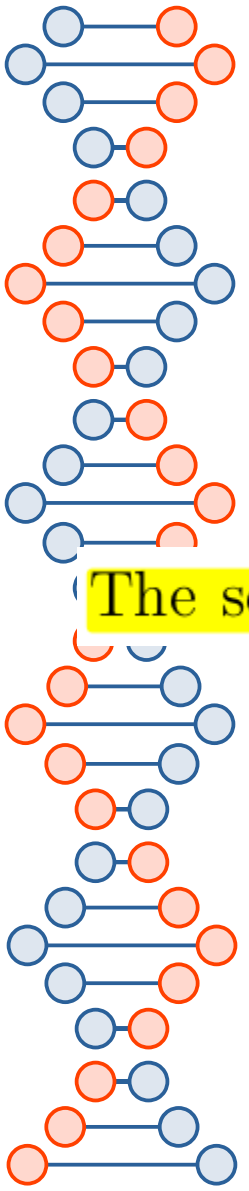


D Computing D in Some Other Way

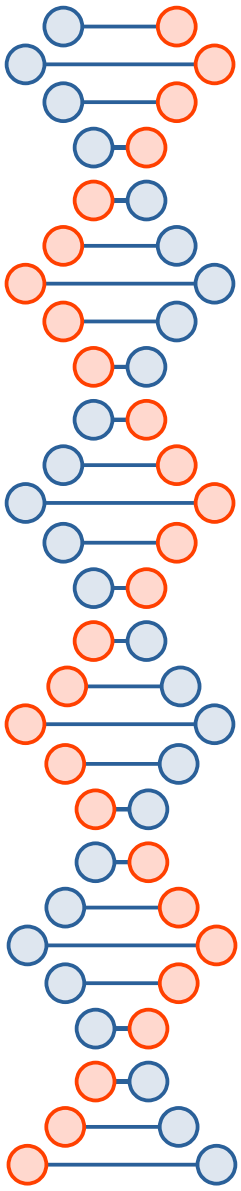
Although this problem of "computing e-th roots modulo n without factoring n" is not a well-known difficult problem like factoring, we feel reasonably confident that it is computationally intractable. It may be possible to prove that any general method of breaking our scheme yields an efficient factoring algorithm. This would establish that any way of breaking our scheme must be as difficult as factoring. We have not been able to prove this conjecture, however.

Our method should be certified by having the above conjecture of intractability withstand a concerted attempt to disprove it. The reader is challenged to find a way to "break" our method.



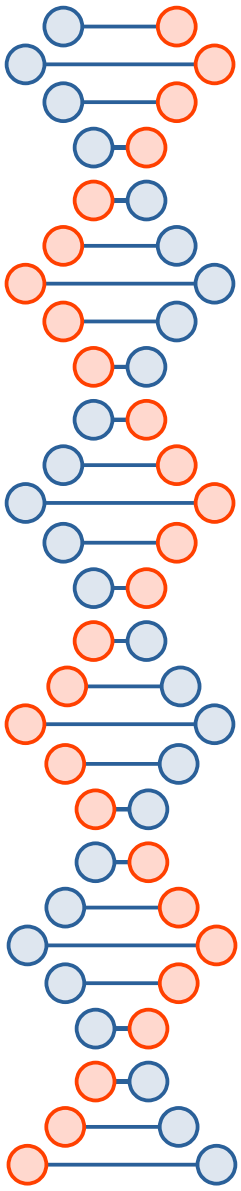


The security of this system needs to be examined in more detail.



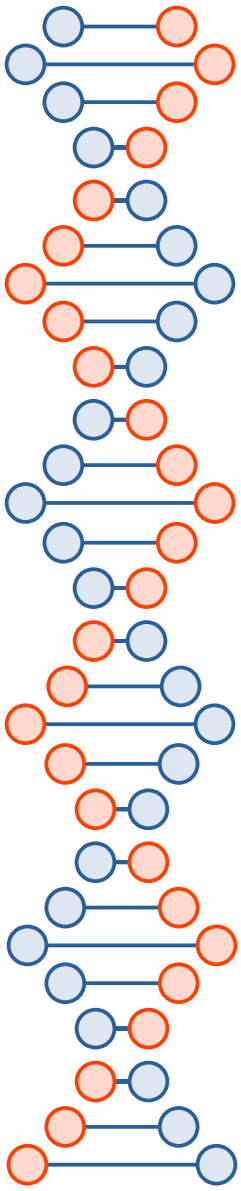
More takeaways

- RSA depends on “we’ve tried to crack it for a long time, but couldn’t”, as does DES, AES, *etc.*
- Textbook RSA is not good enough
- Some differences with Diffie-Hellman
  - Threat model
  - RSA is tricky to implement in a secure way
  - Composite number
  - Who gets to contribute randomness?
- Similarities?
  - Both are broken by quantum computers



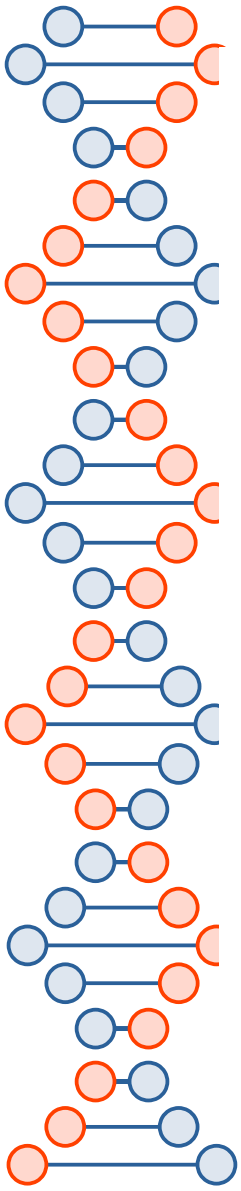
RSA in real cryptosystems

- What we just learned, and read about in the paper, is called “Textbook RSA”
  - Not secure and should not be used (padding is strictly necessary in real schemes)
  - Padding oracle attacks (same idea as for CBC)
- Side channels (covered in a previous lecture)



Side notes

- GCHQ claims to have invented RSA in 1973, and declassified this info in 1997
- In my own research (looking for amateurish crypto in Android apps) using RSA for key distribution is often a red flag
  - An authenticated version of Diffie-Hellman is better, most common thing these days is ECDH (Elliptic Curve Diffie-Hellman)



Let C be the RSA encryption of 128-bit AES key k with RSA public key (n, e). Thus, we have

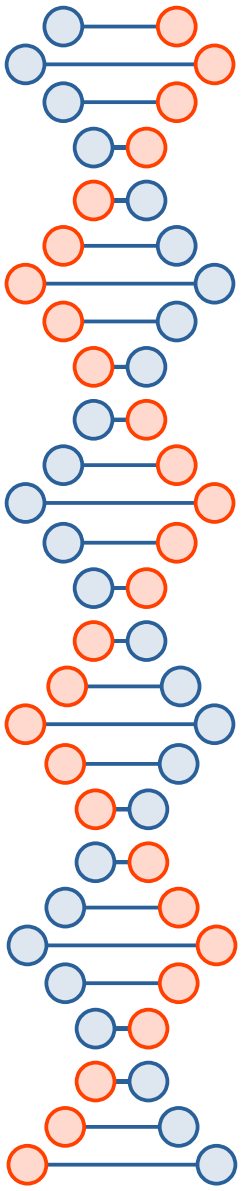
$$C \equiv k^e \pmod{n}$$

Now let C_b be the RSA encryption of the AES key

$$k_b = 2^b k$$

i.e., k bitshifted to the left by b bits. Thus, we have

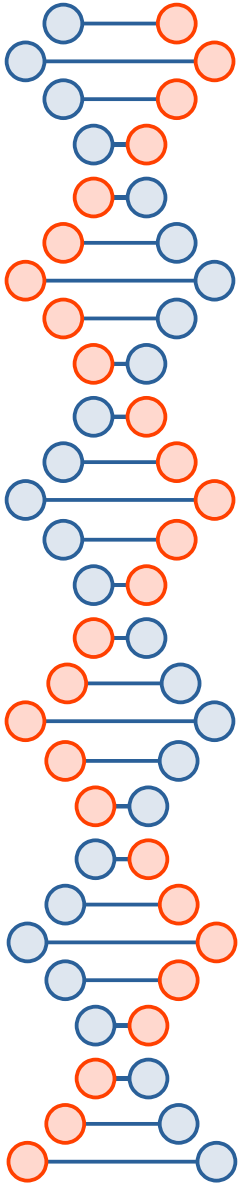
$$C_b \equiv {k_b}^e \pmod{n}$$



$$C_b \equiv {k_b}^e \pmod{n}$$

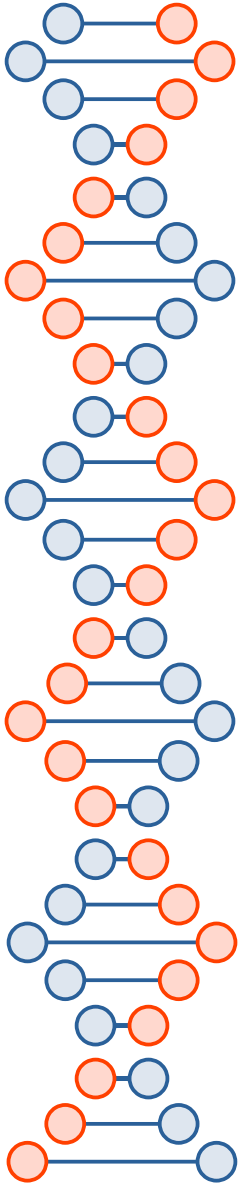
We can compute C_b from only C and the public key, as

$$\begin{aligned} C_b &\equiv C(2^{be} \bmod n) \pmod{n} \\ &\equiv (k^e \bmod n)(2^{be} \bmod n) \pmod{n} \\ &\equiv k^e 2^{be} \pmod{n} \\ &\equiv (2^b k)^e \pmod{n} \\ &\equiv {k_b}^e \pmod{n} \end{aligned}$$



WUP requests

- Full attack is at: https://arxiv.org/pdf/1802.03367.pdf
- The other issues in that paper and previous papers have been fixed, but they still appear to be using textbook RSA



Next lecture

- WUP request attack on RSA in more detail
- Optimal Assymetric Encryption Padding (OAEP)
  - To prevent padding oracle attacks on RSA
- Random oracle model
- Formalizing attacks
  - Ciphertext only, known plaintext, chosen plaintext
  - Chosen ciphertext
  - CPA, CPA2, CCA, CCA2 (2 = adaptive)