Fast Fourier Transform

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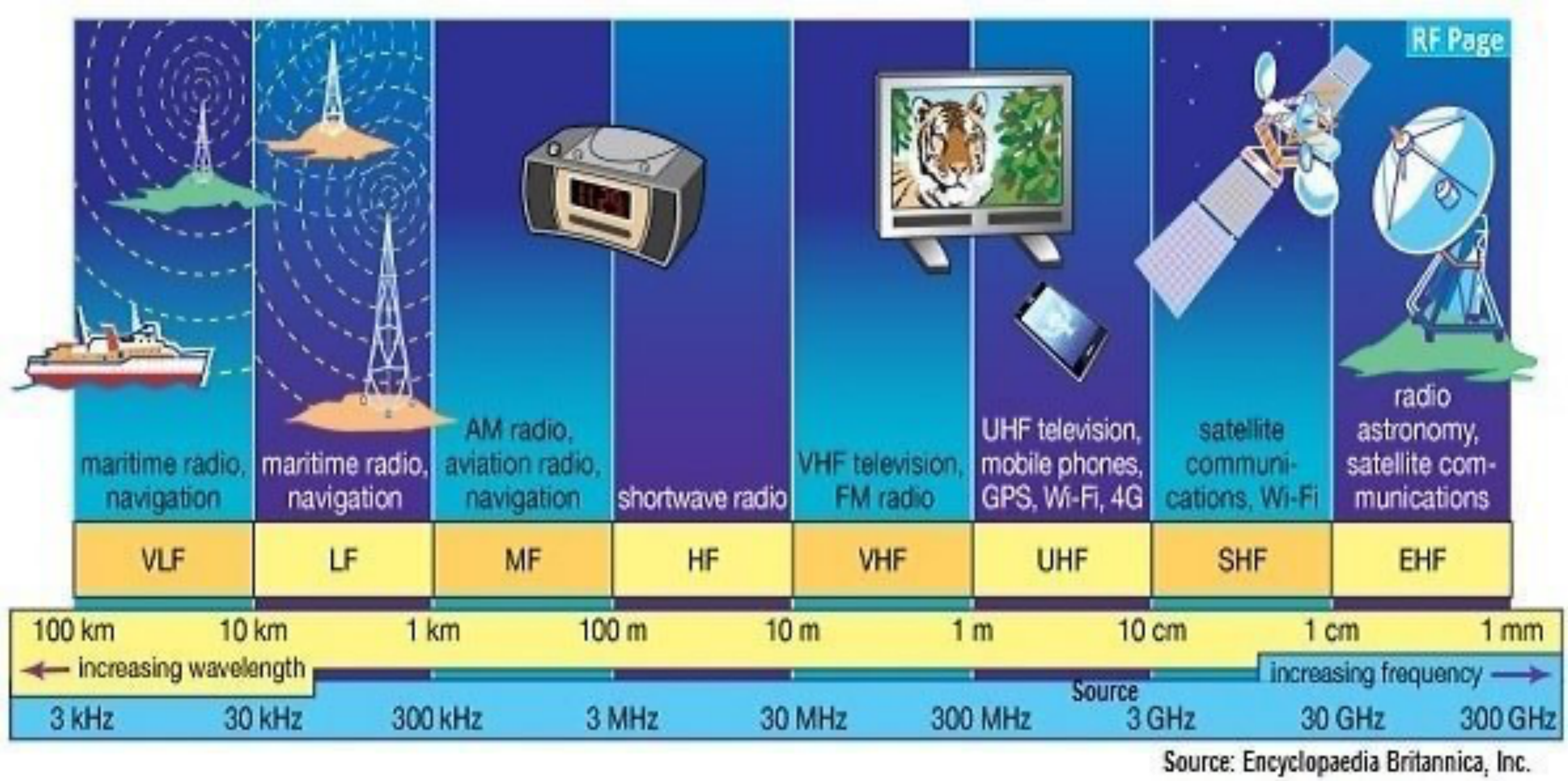
CSE 548



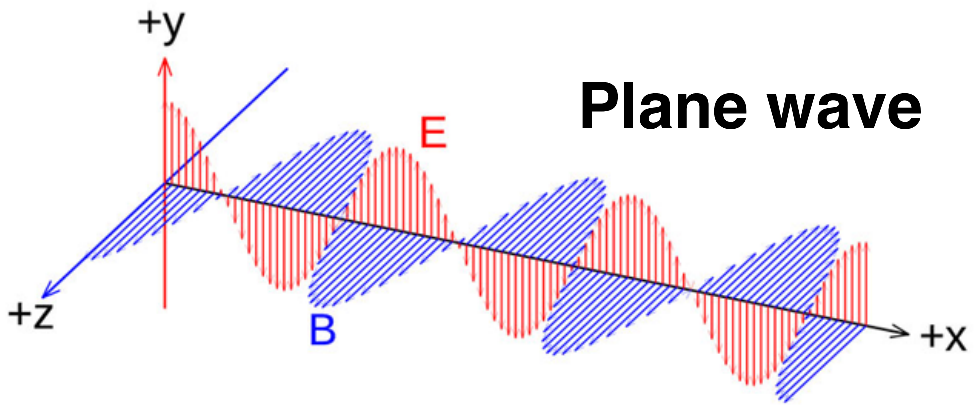
https://www.rfpage.com/what-are-radio-frequency-bands-and-its-uses/#google_vignette

Representing wave as a complex quantity makes math easier.

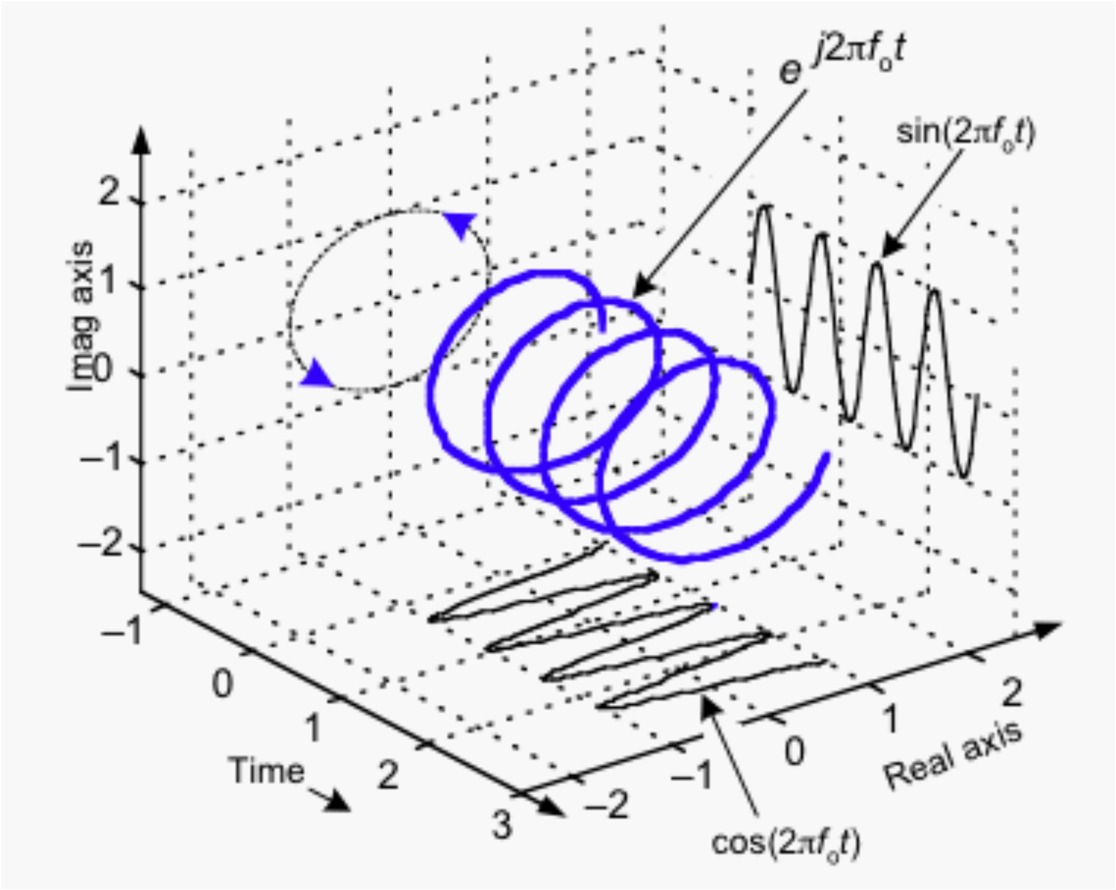
$$\widetilde{E}(x,t) = \quad E_o \exp i\left(\kappa' x - \omega t + \phi\right)$$

$$E(x,t) = \Re\left\{E_o \exp i\left(\kappa' x - \omega t + \phi\right)\right\}$$

Euler's Theorem: $e^{i\theta} = \cos\theta + i\sin\theta$



Plane wave



https://www.oceanopticsbook.info/view/theory-electromagnetism/level-2/plane-wave-solutions#x1-1002r1

https://math.stackexchange.com/questions/144268/is-there-a-name-for-this-type-of-plot-function-on-complex-plane-vs-time-shown

https://phys.libretexts.org/Bookshelves/Quantum_Mechanics/Introductory_Quantum_Mechanics_(Fitzpatrick)/02%3A_Wave-Particle_Duality/2.03%3A_Representation_of_Waves_via_Complex_Functions

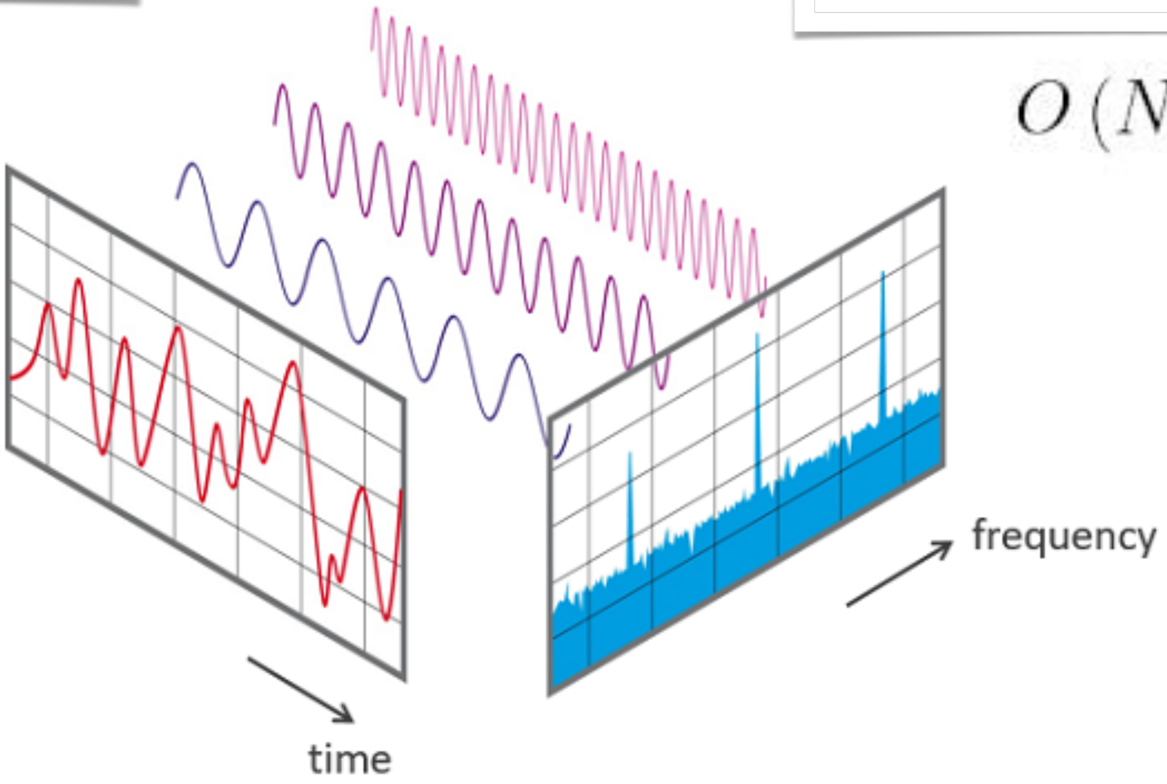
Fourier Transform & Discrete Fourier Transform (DFT)

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$O\left(N^2\right)$

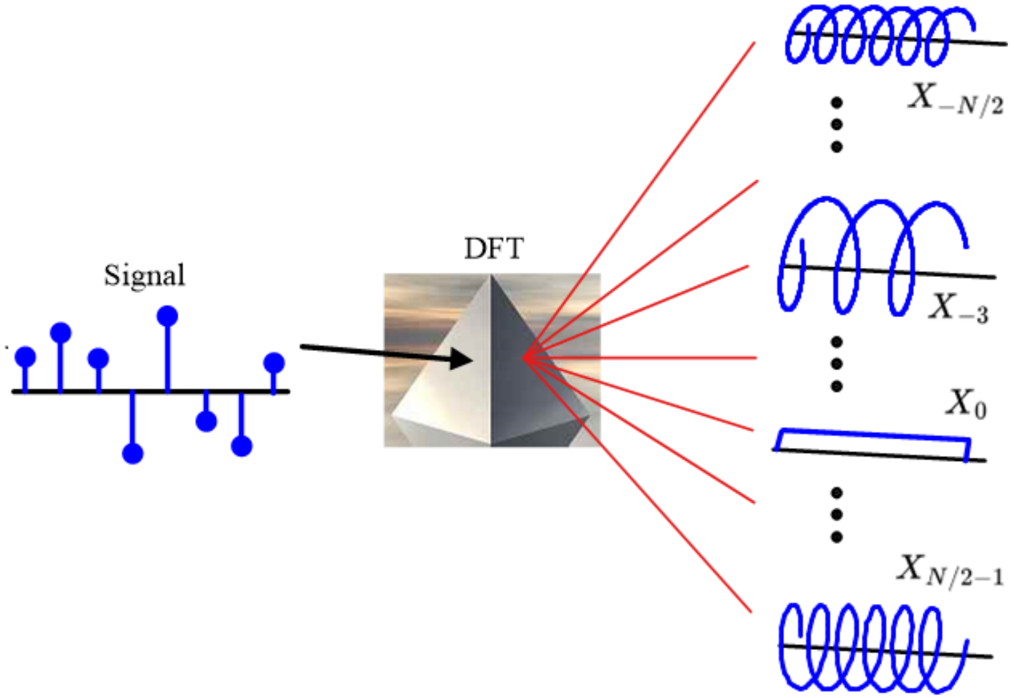
$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

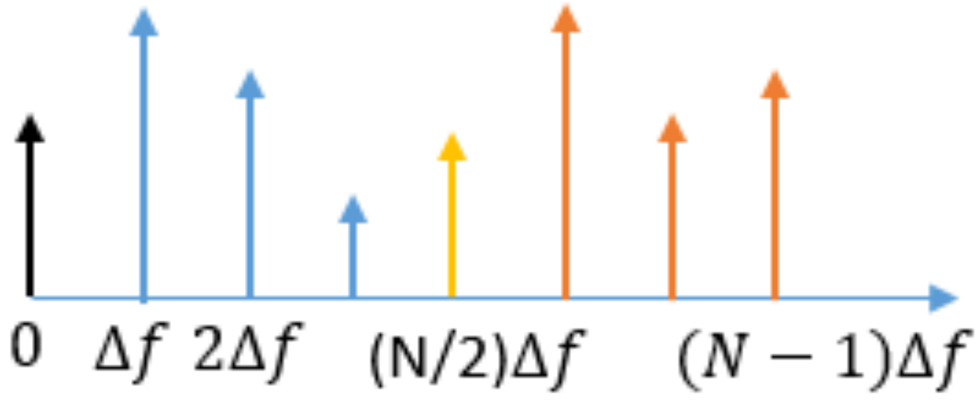
$O\left(N \log_2 N\right)$



https://www.nti-audio.com/en/support/know-how/fast-fourier-transform-fft

DFT





$$\Delta f = \frac{F_s}{N}$$

F_s = Sampling frequency

N = Total number of input samples

Frequency Range $= 0, \ldots, F_s$

https://wirelesspi.com/the-discrete-fourier-transform-dft/

https://www.gaussianwaves.com/2015/11/interpreting-fft-results-complex-dft-frequency-bins-and-fftshift/

FFT

$$\underbrace{X[k]}_{\text{Frequency bins}} = \sum_{n=0}^{\overbrace{N}^{\text{Total Number of Samples}}-1} \underbrace{x[n]}_{\text{Input Samples}} \underbrace{W_N^{nk}}_{\text{Twiddle Factor}}$$

$$X[k] = E[k] + \omega_k \cdot O[k]$$

$$X[k + \tfrac{N}{2}] = E[k] - \omega_k \cdot O[k]$$

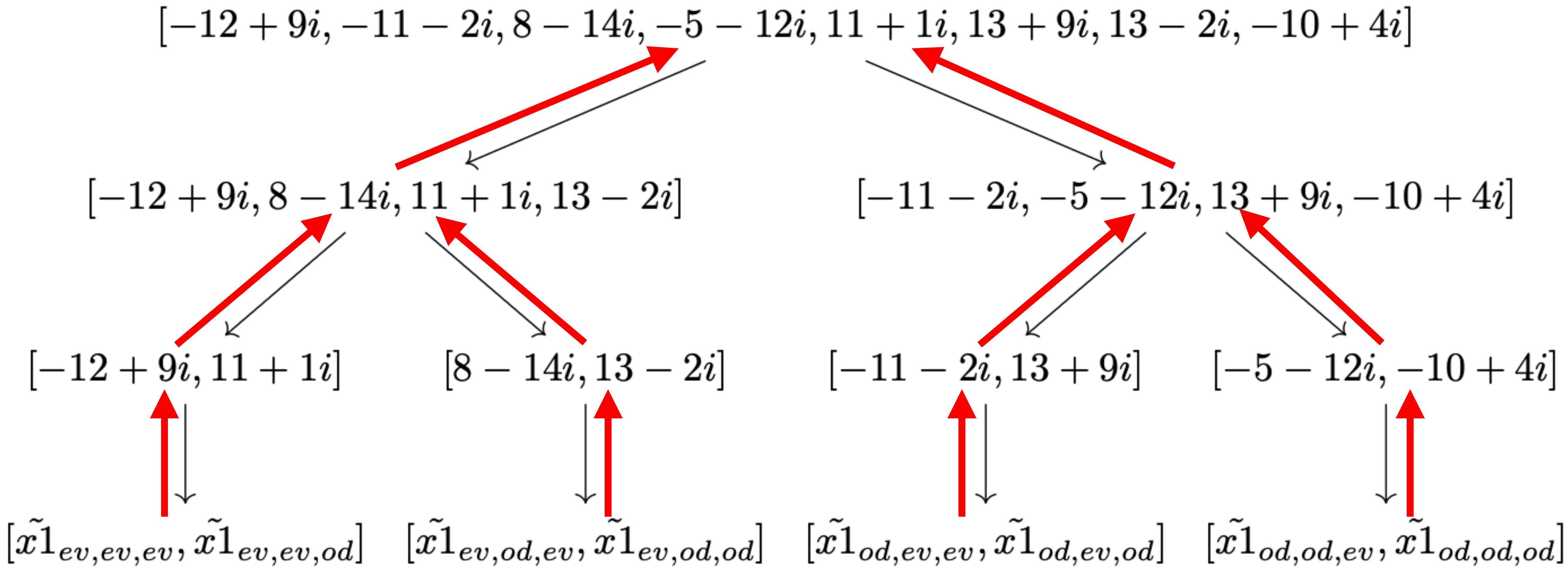
$$k = 0 \text{ to } \tfrac{N}{2} - 1$$

https://www.hesliplabs.com/blog/writing-your-own-fft-for-python-amp-stm32

Cristopher Moore and Stephan Mertens. 2011. The Nature of Computation. Oxford University Press, Inc., USA.

$$[-12 + 9i, -11 - 2i, 8 - 14i, -5 - 12i, 11 + 1i, 13 + 9i, 13 - 2i, -10 + 4i]$$

$$
\begin{tikzcd}[column sep=small]
 & & & {[-12+9i, -11-2i, 8-14i, -5-12i, 11+1i, 13+9i, 13-2i, -10+4i]} \arrow[dll] \arrow[drr] & & & \\
 & {[-12+9i, 8-14i, 11+1i, 13-2i]} \arrow[dl] \arrow[dr] & & & & {[-11-2i, -5-12i, 13+9i, -10+4i]} \arrow[dl] \arrow[dr] & \\
{[-12+9i, 11+1i]} \arrow[d] & & {[8-14i, 13-2i]} \arrow[d] & & {[-11-2i, 13+9i]} \arrow[d] & & {[-5-12i, -10+4i]} \arrow[d] \\
{[\tilde{x1}_{ev,ev,ev}, \tilde{x1}_{ev,ev,od}]} & & {[\tilde{x1}_{ev,od,ev}, \tilde{x1}_{ev,od,od}]} & & {[\tilde{x1}_{od,ev,ev}, \tilde{x1}_{od,ev,od}]} & & {[\tilde{x1}_{od,od,ev}, \tilde{x1}_{od,od,od}]}
\end{tikzcd}
$$


$[-12+9i, -11-2i, 8-14i, -5-12i, 11+1i, 13+9i, 13-2i, -10+4i]$
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$[\tilde{x1}_{ev,ev,ev}, \tilde{x1}_{ev,ev,od}]$
$[\tilde{x1}_{ev,od,ev}, \tilde{x1}_{ev,od,od}]$
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$[\tilde{x1}_{od,od,ev}, \tilde{x1}_{od,od,od}]$

Twiddle Factors

$$\omega_k = e^{-2\pi i k/2}$$

$$\omega_0 : \quad e^{-2\pi i \cdot 0/8} = e^0 = 1$$

$$\omega_1 : \quad e^{-2\pi i \cdot 1/8} = e^{-\pi i/4} = \cos(\tfrac{-\pi}{4}) + i\sin(\tfrac{-\pi}{4}) = \tfrac{1-i}{\sqrt{2}}$$

$$\omega_2 : \quad e^{-2\pi i \cdot 2/8} = e^{-\pi i/2} = \cos(\tfrac{-\pi}{2}) + i\sin(\tfrac{-\pi}{2}) = -i$$

$$\omega_3 : \quad e^{-2\pi i \cdot 3/8} = e^{-\pi i 3/4} = \cos(\tfrac{-3\pi}{4}) + i\sin(\tfrac{-3\pi}{4}) = -\tfrac{1+i}{\sqrt{2}}$$

EXAMPLE